Nonreciprocity in acoustic and elastic materials

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Abstract | The law of reciprocity in acoustics and elastodynamics codifies a relation of symmetry between action and reaction in fluids and solids. In its simplest form, it states that the frequency-response functions between any two material points remain the same after swapping source and receiver, regardless of the presence of inhomogeneities and losses. As such, reciprocity has enabled numerous applications that make use of acoustic and elastic wave propagation. A recent change in paradigm has prompted us to see reciprocity under a new light: as an obstruction to the realization of wave-bearing media in which the source and receiver are not interchangeable. Such materials may enable the creation of devices such as acoustic one-way mirrors, isolators and topological insulators. Here, we review how reciprocity breaks down in materials with momentum bias, structured space-dependent and time-dependent constitutive properties, and constitutive nonlinearity, and report on recent advances in the modelling and fabrication of these materials, as well as on experiments demonstrating nonreciprocal acoustic and elastic wave propagation therein. The success of these efforts holds promise to enable robust, unidirectional acoustic and elastic wave-steering capabilities that exceed what is currently possible in conventional materials, metamaterials or phononic crystals.

Looking at the rippled surface of a pond, we hardly expect water rings to shrink and converge to their centre. It could seem that the growth of water rings with time is an unshakable law of physics; it is not. Under the right circumstances, converging rings can appear, for example, by carefully placing a circular rim on the water surface or, more dramatically, by switching off gravity for a split second[1]. Wave motion usually abides by this principle of time-reversal symmetry: all possible motion, reversed in time, is equally possible. A subtler law of symmetry in wave motion is known as the law of reciprocity (BOX 1). This law states that, in the presence of a vibration source, the signal received by a receiver remains the same when source and receiver are interchanged. For instance, two water striders floating on the surface of the pond are equally vulnerable to the ripples generated by the other: if the first shakes its legs with a given force sending ripples that displace the water under the second by a given amount, the second can reciprocate by displacing the same amount of water under the first by applying the same force. What is remarkable about these two symmetries, time reversal and reciprocity, is that they are typically oblivious to composition and geometry. In particular, they hold in the presence of any amount of elastic scattering and reflection by arbitrarily shaped heterogeneities and boundaries. Accordingly, they have been leveraged to enable a number of widely applicable techniques in experimental characterization and numerical modelling, such as time-reversed acoustics and the boundary-element method[2–9]. Perhaps more extraordinary is that, although the wave equation is not invariant under time reversal in the presence of dissipation, as is the case in viscous fluids, the fundamental property of reciprocity remains unchanged[10,11].

Reciprocity can also be seen as a hindrance. For instance, under reciprocity, there is no way to tune transmission to two different levels in opposite directions. As a result, the creation of acoustic and elastic wave devices that exploit unidirectional transmission, such as acoustic diodes[12], is impossible in the presence of reciprocity. The ability to create materials and systems that enable nonreciprocal wave transport in acoustic and elastic media is, therefore, of substantial interest in broad areas of engineering and science[13,14]. The purpose of this Review is to survey designs and strategies that overcome this fundamental limitation of classical acoustic and phononic systems, and enable purposeful and tunable nonreciprocal wave propagation.

Reciprocally exchanged vibrations do not only have the same amplitude but also identical time profiles.

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In particular, their spectra feature the same frequency and phase content. Accordingly, the frequency shift observed in the Doppler effect is fundamentally nonreciprocal. Indeed, if, in the original configuration, a source communicates to a receiver through a barrier moving with the sound, then, in the flipped configuration, the barrier moves against the sound. This means that the Doppler shift changes signs when the source and receiver are swapped, contradicting reciprocity. We should, therefore, expect reciprocity to fail in systems with mean flow[15] or external biasing, such as magnetic fields applied to ferromagnetic materials[16] or Coriolis forces[17], all of which break time-reversal symmetry. Time reversibility and reciprocity are physical analogues, but their opposites are not necessarily equivalent. Examples of time-irreversible mechanisms include both external biasing and energy-dissipation effects such as viscosity. Both are associated with a loss of time reversibility, but in physically different ways. In particular, reciprocity is maintained in linear acoustics with viscous damping (BOX 2). More generally, we anticipate the failure of reciprocity in the presence of mechanisms that generate harmonics, such as nonlinear and time-dependent constitutive material properties.

This Review is organized around four research areas that have been explored to create nonreciprocal materials offering directional wave control: kinetic media (media with moving parts or circulating flows); activated media (media with time-dependent properties); topologically protected one-way edge states (waves localized at the boundaries of some kinetic or activated media); and nonlinear media (media displaying nonlinear elastic behaviour).

It is worthwhile to stress that spatial symmetry and reciprocity, while interdependent, are fundamentally different (BOX 3). Reciprocity is assured in configurations in which the source and receiver are interchangeable under spatial symmetry. Conversely, spatial asymmetry does not imply nonreciprocity, but it can enable it. Spatially asymmetric wave propagation can be useful, be it reciprocal or not, and has been increasingly investigated in recent years in an emergent class of media known as topological insulators. These are materials that stop waves from propagating through their bulk, while favouring their transmission along edges and interfaces in specific directions. The interplay between spatial asymmetry and nonreciprocity in topological insulators constitutes one of the thrusts of the present Review. Another thrust is the physical consequences of spatial asymmetry combined with nonlinear material properties: the departure from linearity opens up the possibility of nonreciprocal dynamics in passive media, although the breaking of spatial mirror symmetry is a necessary and essential ingredient.

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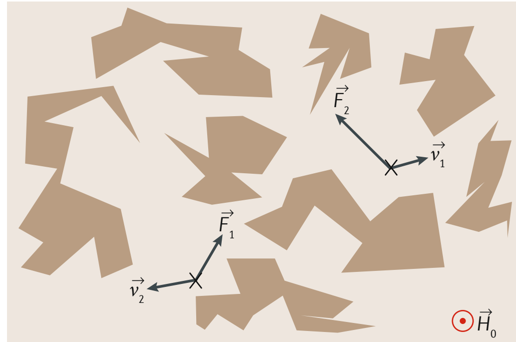
Nonreciprocity in kinetic media

All linear time-invariant wave systems are inherently reciprocal, unless a quantity H_0, external to the system, biases the motion by breaking time-reversal symmetry on a microscopic scale (BOX 2). To do so, H_0 must be time odd, that is, it must flip sign under time reversal. For instance, elastic waves in ferromagnetic crystals can be nonreciprocal in the presence of an external magnetic field, although magnetoelastic phenomena tend to be very weak in natural materials[16].

A simple example of a nonreciprocal linear time-invariant medium is a fluid in constant motion with velocity v_0 smaller than the speed of sound c_0. The motion, which is imparted by an external system (such as a pump) independently of the presence or absence of sound in the medium, imparts a mean momentum to the acoustic medium. This momentum can be interpreted as a time-odd bias. If one imagines two points placed along the flow, it is clear that the speed of sound along the flow, $c_0 + v_0$, differs from that against the flow, $c_0 - v_0$, thus, the transmitted signals along or against the flow differ by a phase (assuming the absence of losses, BOX 4). This simple idea can be exploited to build acoustic gyrators[18] in air pipes under steady flow and isolators[19]. If the length of the pipe is properly adjusted, waves excited along the flow accumulate a phase of 90°, whereas in the opposite direction, the phase is −90° (FIG. 1a). A 180° phase difference between forward and backward waves occurs for this specific waveguide length, for which the system acts as a gyrator (FIG. 1b). This example allows us to point out another challenge in building nonreciprocal systems: because the bias v_0 is generally small compared with c_0, appreciable nonreciprocal transmission phases occur only after long distances, namely, for systems much larger than the wavelength. Using grating elements to slow down sound, it is possible to shrink the size of nonreciprocal devices and realize compact isolators and gyrators for guided and radiated waves[19].

Interestingly, the effect of fluid motion on acoustic wave propagation can lead to similar phenomena as those observed for electrons in a magnetic field. For instance, an acoustic beam passing through a vortex acquires a nonreciprocal phase shift analogous to the Aharonov–Bohm phase for an electron going through a region of space with non-zero magnetic vector potential[20]. An acoustic analogue of the Zeeman effect (the splitting of the electronic energy levels in an atom subject to a magnetic field) has also been demonstrated in a ring resonator subject to constant rotating air flow[15]. This design was then exploited to construct an acoustic circulator whose resonant nature allowed the use of slow fluid

Box 1 | Reciprocity



Early hints of reciprocity can be found in 'Mécanique Analytique' of Joseph-Louis Lagrange[175], as noted by Horace Lamb[176]. Afterwards, important particular cases were dealt with by Hermann von Helmholtz[177] in an 1860 paper on sound in tubes and by Alfred Clebsch[178] in his work of 1862 on systems of rods. Clebsch noted that the coefficients of the dynamical matrix relating nodal displacements to external loads are symmetric, although he did not formulate that property as a statement of reciprocity. This was later done by James Clerk Maxwell[179] while investigating the same topic in 1864. One of Maxwell's theorems reads, for four arbitrary points (B, C, D, E) in a continuous medium: "The extension in BC, due to unity of tension along DE, is always equal to the tension in DE due to unity of tension in BC." Enrico Betti extended the concept of reciprocity to the work done by static forces on an elastic body in 1872 (REF.[180]). In acoustics and elastodynamics, a reciprocal theorem, namely, that "the vibration excited at A will have at B the same relative amplitude and phase as if the places were exchanged," was stated and proved by Lord Rayleigh[181] in 1873. He seems to be the first to refer to the 'reciprocal character' of the property. Rayleigh appreciated the extent and generality of the reciprocal theorem and considered the presence of heterogeneities and of linear damping. The reciprocity theorems of Maxwell and Rayleigh are concerned with cases where the forces are concentrated at a single member or point, whereas Betti's theorem is more general, in that it considers distributed forces. Various equivalent mathematical formulations of reciprocity exist[182,183]. The reciprocity theorem applies equally to media with linear mechanisms for wave attenuation, including viscous loss[10,11], and media with distributed[183] or moving loads[184].

To provide a concrete example, we show how reciprocity is defined in linear scalar acoustics. We start from the general dynamic equations for an isotropic acoustic medium, written in the frequency domain as $\nabla p = -j\omega\vec{\mu} + \vec{f}$ and $\nabla \cdot \vec{v} = j\omega\varepsilon_v + q$, where j is the imaginary unit, ω the radian frequency, p the acoustic pressure, $\vec{v}$ the particle velocity, $\vec{\mu}$ the momentum density, ε_v the volume strain, and q and $\vec{f}$ are the volume source and body force source distributions, respectively. The constitutive relations are assumed to be of the Willis type, namely, $\vec{\mu} = \underline{\rho} \cdot \vec{v} - \vec{\eta}\, p$ and $\varepsilon_v = -\beta p + \vec{\gamma} \cdot \vec{v}$, where β is the compressibility, $\underline{\rho}$ the second-order dynamic density tensor, and $\vec{\eta}$ and $\vec{\gamma}$ are Willis coefficients that couple momentum density to time-varying pressure and volume strain to velocity, respectively[58]. These relations for time harmonic motion are based on the assumption that the medium is time-invariant, that is, that its constitutive properties $(\underline{\rho}, \vec{\eta}, \vec{\gamma}, \beta)$ are independent of time. Reciprocity links two distinct excitation situations 1 and 2, in which the same medium is considered but the source distributions $q_{1,2}$ and $\vec{f}_{1,2}$ are different, leading to distinct field solutions $p_{1,2}$ and $\vec{v}_{1,2}$. More specifically, reciprocity is defined[11] for an infinite medium as the property $<1,2> = <2,1>$, where $<1,2> = \iiint \mathrm{d}V(\vec{f}_1 \cdot \vec{v}_2 - q_1 p_2)$. Using the dynamic equations and the constitutive relations, it is straightforward to apply Gauss' theorem and show that, in general, $<1,2> - <2,1> = -j\omega \iiint \mathrm{d}V[(\vec{\eta} + \vec{\gamma}) \cdot (p_1\vec{v}_2 - p_2\vec{v}_1) + \vec{v}_1 \cdot (\underline{\rho} - \underline{\rho}^{\mathrm{T}}) \cdot \vec{v}_2]$. Note that this assumes a finite source distribution and infinite volume of integration, which eliminates surface contributions. This relationship shows that, in the absence of Willis coupling ($\vec{\eta} = \vec{\gamma} = \vec{0}$), reciprocity holds if the density is symmetric, $\underline{\rho} = \underline{\rho}^{\mathrm{T}}$, whereas in the presence of Willis coupling, it holds only if, additionally, $\vec{\eta} = -\vec{\gamma}$.

Similar restrictions imposed by causality on the stiffness, coupling and density tensors in the context of Willis solids were derived[185], namely, that breaking the major symmetries of any of those material-property tensors violates the requirements for reciprocal wave motion. In the present example, the creation of acoustic media with $\underline{\rho} \neq \underline{\rho}^{\mathrm{T}}$ or Willis media for which $\vec{\eta} \neq -\vec{\gamma}$ are, therefore, two different ways to break reciprocity. As further discussed in BOX 2, the relation $\vec{\eta} = -\vec{\gamma}$ is typically satisfied by most media, as it follows directly from the Onsager–Casimir relations, which generally hold for linear time-invariant systems in the absence of external time-odd bias.

The figure illustrates the reciprocity theorem in an inhomogeneous domain consisting of a background medium containing heterogeneities (darker regions). Specifically, it shows point-like source forces (dipoles), $\vec{F}_1$ and $\vec{F}_2$, applied at different locations. In this case, the reciprocity condition $<1,2> = <2,1>$ is equivalent to the equality of the virtual powers, namely, $\vec{F}_1 \cdot \vec{v}_2 = \vec{F}_2 \cdot \vec{v}_1$. In linear, time-invariant acoustic and elastic media, the theorem can only be invalidated if time-reversal symmetry is broken at a microscopic scale. This can be achieved by introducing a momentum in the wave-bearing medium, represented by $\vec{H}_0$, that is, through the use of a mean flow field that does not change direction under time reversal[186]. The only other way to violate reciprocity is to break the linearity or time-invariance of the medium itself.

motion to achieve large nonreciprocal behaviour (FIG. 1c). At first, the cylindrical acoustic cavity, coupled to three external waveguides, contains air at rest. Consequently, reciprocity holds and the scattering matrix is symmetric (waves incident at a given port split equally between the other two ports). By properly rotating the air filling the cylindrical cavity, it is possible to strongly break reciprocity and create an acoustic circulator. This idea was validated in a series of experiments using airborne audible sound and air motion implemented through standard fans.

Nonreciprocity in activated materials

Background. Activated materials modulate their constitutive properties in space and time in response to an external stimulus. These materials violate time-invariance and can, thus, exhibit a wide array of nonreciprocal wave phenomena. For example, in an activated two-phase laminate, as time passes, the spatial phase profile is translated at constant speed (FIG. 2a); other phononic crystals are activated by varying the properties of a unit cell as a function of time (FIG. 2b). Rather than modulating bulk properties, an activated structure can also be obtained by modulating an interaction coefficient between the structure and, for instance, attached resonators or the ground (FIG. 2c), or, more generally, by modulating any boundary condition. In acoustics, metamaterials with time-dependent effective properties may be obtained from time-modulated meta-atoms built from electro-acoustic transducers, which have been successfully used to provide various advanced static functionalities, such as nonlinearity[21,22] and acoustic gain[23], and can, in principle, be reconfigured faster than the acoustic wave[24].

From a modelling perspective, wave motion in activated materials is described by equations with time-dependent constitutive parameters, such as the wave equation $c^2(x,t)\nabla^2\phi(x,t)=\ddot{\phi}(x,t)$, where the modulated physical property c, the speed of sound, depends on both space and time variables. The emphasis here is on modulations in the form of progressive periodic waves, referred to as pump waves, for example, $c(x,t)=c_o+\tilde{c}(x-vt)$ where $\tilde{c}$ is periodic. Such progressive modulations create a space–time bias that enables nonreciprocal wave motion as a function of the modulation frequency f_M and depth, $|\tilde{c}|/c_o$ (FIG. 2d). Small-amplitude, medium-speed modulations lead to a Bragg scattering regime; faster modulations can be described by an equivalent medium with effective properties; and slower but higher-amplitude modulations lead to an adiabatic regime.

Box 2 | Principle of microscopic reversibility and reciprocity

At first glance, reciprocity may appear to be a consequence of time-reversal symmetry in the propagation medium. Indeed, when all terms in the wave equation are invariant under time reversal, solutions come in time-reversed pairs; one for t and another for $-t$. This means that, if a wave can propagate in one direction, the time-reversed wave, which propagates in the opposite direction, is also a viable physical solution. However, reciprocity is a more fundamental symmetry of the response function between two points in a medium. It compares the displacement or velocity at point B due to a perturbative force applied at point A to that obtained at point A when the same perturbation is imparted at point B. Specifically, a medium is reciprocal if the response functions α_{ik} (also known as generalized susceptibilities) governing the relation $x_i=\alpha_{ik}f_k$ between a quantity x_i and a dual perturbation f_k, potentially applied at a different location, satisfy the symmetry property $\alpha_{ik}=\pm\alpha_{ki}$. Note that the plus and minus signs apply respectively to cases in which the susceptibility relates two quantities that are of the same or opposite parity upon time reversal (for example, the Willis coupling coefficients in BOX 1 couple momentum, which is odd under time reversal and strain, which is even under time reversal; therefore, they must obey $\vec{\eta}=-\vec{\gamma}$).

Lars Onsager[187] and Hendrik Casimir[188] proved that a linear time-invariant medium, slightly perturbed away from thermodynamic equilibrium, and whose governing equations at a microscopic scale obey time-reversal symmetry is necessarily a reciprocal medium. The latter hypothesis is known as the principle of microscopic reversibility. To illustrate this principle, consider the following example. Suppose that heat conduction in an anisotropic solid obeys Fourier's law $q_i=-k_{ij}\partial_j T$, where $\vec{q}$ is the local heat flux density vector, k the thermal conductivity of the material and T the temperature. It is well known that heat conduction is macroscopically irreversible: a temperature evolution reversed in time is not physically admissible, as it contradicts the second law of thermodynamics. However, heat conduction is microscopically reversible, because the equations governing the motion of particles on the microscopic scale obey time-reversal symmetry. Consequently, in the linear regime, heat conduction is reciprocal and $k_{ij}=k_{ji}$. Note that Rayleigh's reciprocity theorem, in its historical form, admits simple, direct proofs for the propagation of acoustic fields at the macroscopic scale, but can readily be obtained from the Onsager–Casimir theorem of reciprocity by letting x_i be a component of the particle velocity and f_k a component of the volume force. One can break reciprocity by lifting one or several of the assumptions behind the Onsager–Casimir theorem, such as linearity, time-invariance and/or microscopic reversibility. For instance, microscopic reversibility is invalid in the presence of a magnetic field, Coriolis forces or of a mean flow. In each of these examples, a microscopic momentum bias, which does not change on time reversal, is imparted at the microscale, leading to a violation of reciprocity for macroscopically observable fields.

The possibility of locally breaking the Onsager–Casimir relations, as in fluids with odd viscosity[189], biased piezoelectric composites[190] or non-reciprocal Willis media[55,191], demonstrates that one does not necessarily need to compare transmission between a sender and a receiver to see the nonreciprocal effect, which is already apparent in the local constitutive relations of the medium. In summary, reciprocity is separate from and independent of the general notion of time-reversal symmetry. Reciprocity holds in the presence of material dissipation, including viscous losses, which is not time reversible[11,182], but it fails under conditions of mean flow or magnetic forcing in magnetosensitive materials.

One-way Bragg mirrors. Waves incident on a phononic crystal are scattered as they encounter geometric irregularities, boundaries and changes in material properties. Simply speaking, incident wavefronts are partially reflected by periodically spaced objects with acoustic impedance that differs from that of the background medium. What is remarkable is that, even when individual objects reflect weakly, strong reflection from the ensemble, known as Bragg reflection, can be observed under a condition of constructive interference dependent on the wavelength and angle of incidence. Conversely, destructive interference means that the wave is not effectively reflected and is transmitted unaltered. Either way, if the crystal is stationary, the incident, transmitted and reflected waves all have the same frequency. By contrast, a Doppler-like effect occurs when the crystal has properties that are modulated in space and time by the action of a pump wave. In this case, the frequency of the reflected wave is shifted up or down, depending on the relative motion of the incident wave and the pump wave[25]. The Doppler shift, in turn, modifies the condition for constructive interference, which acquires an extra parameter: the sense of propagation. A one-way Bragg mirror[26–28] can, therefore, be conceived by tuning the Doppler shift to favour reflection when the incident and pump waves propagate, say, in the same direction, and favour transmission when they propagate in opposite directions. Other anomalous Doppler effects may be observed[29] in a periodic acoustic medium when the sound source is moving. Thus, sound from a stationary source at a frequency inside the bandgap of the periodic medium cannot be heard by an observer in the far field (even if the observer moves), whereas the sound from a moving source can.

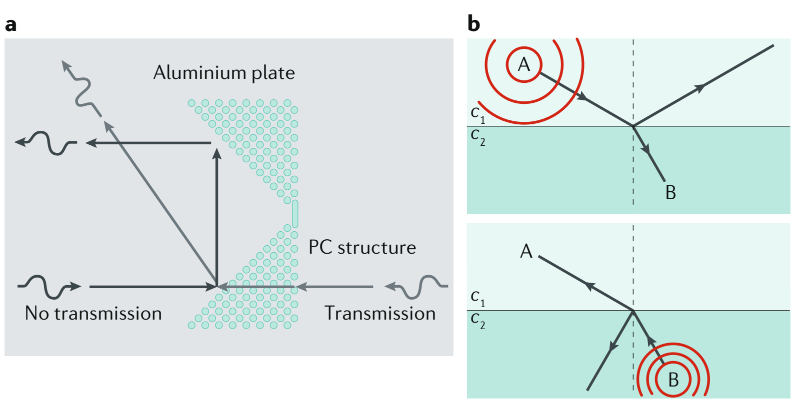
The first experimental evidence[30] of nonreciprocal Bragg scattering in activated materials was observed in a structure of ring magnets sliding over a common axial rail and housed by grounded solenoids (FIG. 3a). Each ring repels or attracts its host solenoid by a force proportional to the current input of the solenoid. By modulating the current with a frequency f_M and a wavenumber q_M, the structure effectively behaves as a grounded spring–mass chain with a modulated grounding stiffness $k_g(x,t)=k_o+\delta k\sin(q_M x-2\pi f_M t)$. Inspection of the frequency-response function shows that reciprocity breaks down at select frequencies $f_{i,j}$ at which the modulation favours co-propagated signals over counter-propagated ones (FIG. 3b). These frequencies satisfy the Bragg condition $f_i-f_j=nf_M$, where n is an integer.

Box 3 | **Asymmetric and nonreciprocal wave propagation**

It is useful to distinguish between asymmetric and nonreciprocal wave propagation because they are not generally equivalent. In systems with asymmetric wave propagation, the monitored output changes if the locations of the input and output are interchanged. In this sense, nonreciprocity can lead to asymmetric wave propagation, but the inverse is not necessarily true. For example, it is possible, in a reciprocal system, to excite an elastic wave at a source location and obtain a different wave type at the receiver owing to a linear mode-conversion mechanism[192]. This process, although intriguing and useful[192–196], is reciprocal. Care should also be taken in the choice of the input and output parameters when checking for reciprocity. As a general rule, the input and output variables to be interchanged while verifying reciprocity should be dual, for example, their product should yield a power or an energy[197]. Thus, reciprocity ensures the invariance of the ratio of the output velocity to the input force; it does not say anything about the ratio of the input and output velocities or forces.

The reciprocity relation takes a statistical form (ensemble average) when it relates the radiation impedance of an elastic structure to the diffuse sound field generated by it[198]. Acoustical reciprocity does not hold if the medium is in motion when the propagation direction has some component orthogonal to the direction of mean flow, such as one may experience in a windy environment[199]. Loss of reciprocity was noted explicitly by Philip Morse and K. Uno Ingard when discussing the refraction of sound between two fluid media with differing mean flow velocities parallel to their interface[200]. For waves in a steady flow, reciprocity is restored[176] if the direction of the flow is reversed when the source and receiver are interchanged; this is also known as the flow-reversal theorem[186]. Another interesting scenario is when the boundary or supports of a linear acoustic medium move. The boundary conditions (the constraint equations) in a system with a moving boundary or support are explicit functions of time. Reciprocity does not hold under these conditions[201]. As a case in point, a spatially asymmetric waveguide with a prescribed harmonic displacement input can exhibit asymmetric end-to-end wave transmission[202]. Some relevant misconceptions regarding nonreciprocity and asymmetric propagation were addressed in the context of acoustic diodes and isolators[12].

The figure shows examples of reciprocal but asymmetric wave transmission. Panel **a** is a schematic representation of linear asymmetric wave propagation in a plate with systematically arranged holes[193]. A wave incident from the left is reflected back (180°) after interacting with the two prismatic arrays of scatterers, whereas a wave incident from the right is transmitted with a 45° change in the direction of propagation. Panel **b** shows the wave transmission between two media with different speeds of sound, $c_1 > c_2$: whereas transmission at the interface can be highly asymmetric in a global sense, it remains reciprocal, as can be proved by considering the point-to-point signalling properties between points A and B. Panel **a** adapted with permission from REF.[193], AIP. PC, photonic crystal.



In a second experimental demonstration[31], the solenoids were supported on a beam substrate using pairs of cantilevers. In this configuration, each solenoid oscillates coaxially with a permanent magnet directly bonded to the host beam. The structure is activated by electrically driving the magnetic coupling between the solenoids and the permanent magnets (FIG. 3c). This configuration allows direct measurement of the Doppler shift by comparing the transmission and reflection spectra (FIG. 3d). A dip in transmission at a frequency f_1 is accompanied by a peak in reflection at a frequency f_2 such that $f_1 - f_2 = f_M$. Thus, the structure operates as a one-way mirror at the pair (f_1, f_2): if frequency f_1 is transmitted when incident from the right, it is reflected into f_2 when incident from the left and vice versa.

A closer look shows that a full, however, small, range of frequencies centred on f_1 and f_2 behave similarly. These define the mirror bandgaps, that is, the frequency bands over which signals cannot penetrate into the mirror. A one-way mirror is easily recognized through the band diagram, which features directional bandgaps that do not extend over the whole Brillouin zone and are restricted to regions of positive or negative incidence (FIG. 3e). Maximum one-way reflection is obtained when the directional bandgaps do not overlay, that is, when the Doppler shift is larger than the bandwidth of the gap.

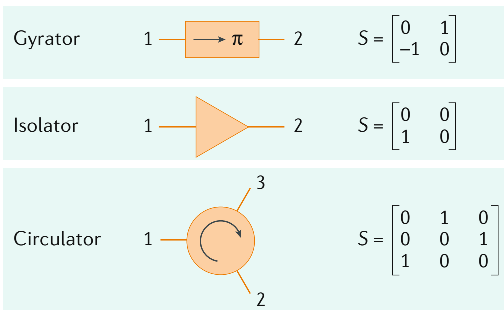
When the background supports multiple propagating modes, as in dispersive waveguides and metamaterials, the frequency shift from one-way reflection can drastically modify the incident mode. For instance, in a resonant metamaterial, an incident acoustic mode can be directionally reflected into an optical mode[32,33]. Furthermore, one-way mode conversion can be triggered in transmission[34–37], although such nonreciprocal transitions are yet to be observed. Other platforms for activated nonreciprocity include 1D piezoelectric structures with space–time-modulated electrical-boundary conditions[38,39], linear piezophononic media under electrical bias[40] and media with space–time-modulated effective mass[41] or effective stiffness[42]. Space–time modulation of related material properties can lead to other nonreciprocal effects: for instance, a travelling-wave modulation of thermal conductivity and specific heat capacity causes the heat flux to have different properties when it propagates in the same or opposite direction to the modulation[43].

Nonreciprocal Willis coupling. Pump waves with larger amplitudes cause stronger scattering; propagated modes no longer contain a handful of components, but, rather, a complete spectrum of frequencies and wavenumbers. In this case, either full Floquet–Bloch analysis or time-domain simulations can be used to construct the band diagrams[44–46]. Simply put, the spatial profile of the modulation dictates the band structure; each band is then sheared, or tilted, by an angle that is a function of the modulation frequency[47–49]. Therefore, large-amplitude fast modulations lead to directional gaps substantially wider than in the case of weak scattering. The consequence is that one-way total reflection can be ensured by modulating smaller regions. Conversely, however, one-way total transmission is no longer possible owing to the large time-varying impedance mismatch between the modulated region and the background.

Beyond directional gaps, strong scattering breaks reciprocity and time-reversal symmetry within passing bands as well. Indeed, the modulation-induced tilt favours, via acceleration, waves propagated either along or against the pump wave, while opposing their time-reversed version through deceleration (FIG. 3f). Taken to extremes, supersonic modulations can ‘freeze’ propagation in a given direction, or even reverse it (FIG. 3g).

The modulated medium then appears to lack modes propagated in a given direction, while exhibiting an excess of modes propagated in the opposite direction[50,51]. This phenomenon is fundamentally different from one-way Bragg reflection, but can still be leveraged for the purposes of one-way mirroring, especially at low frequencies, where wide bandgaps are seldom available (FIG. 3h).

Box 4 | **Gyrators, isolators and circulators**



Many nonreciprocal mechanical systems are built on analogies with multi-port electronic devices. The reciprocity theorem applied to a multi-port acoustical system in which the inputs are x_i and the outputs f_k implies that the scattering matrix S_{ik} ($x_i = S_{ik} f_k$) is symmetric: $S_{ik} = S_{ki}$. The most common linear devices that violate the reciprocity condition are gyrators, isolators and circulators. These systems are, therefore, described by an asymmetric scattering matrix $S_{ik} \neq S_{ki}$. Gyrators and isolators have two ports, whereas circulators have three ports. Let us first consider a general two-port scattering system, described by a two-by-two scattering matrix $S = [S_{11}, S_{12}; S_{21}, S_{22}]$ and ask whether there can be nonreciprocity in a lossless two-port system. Because energy conservation implies unitarity for S, one can use the general parameterization of two-by-two unitary matrices, defining the angles θ, α, ϕ and Φ and write:

$$S = \begin{pmatrix} e^{i\phi}\cos\theta & e^{i\alpha}\sin\theta \\ -e^{-i\alpha}\sin\theta e^{i\Phi} & e^{-i\phi}\cos\theta e^{i\Phi} \end{pmatrix}$$

This form of the scattering matrix implies that the off-diagonal coefficients can differ in phase, but never in amplitude. This means that lossless two-port scattering systems can only be nonreciprocal in the transmission phase. Gyrators are two-port nonreciprocal systems that transmit the same power between their two ports, but achieve the maximum nonreciprocal phase difference of π between S_{12} and S_{21}. However, when one wants to create a one-way propagation device for waves, one needs a nonreciprocal response in the transmission amplitude.

Bernard Tellegen proved that any nonreciprocal response, in phase or amplitude, can be achieved with combinations of gyrators and other reciprocal elements[203]. In particular, to achieve nonreciprocity in amplitude in a two-port system, we must simply relax the assumption of unitarity. For instance, ideal isolators correspond to a subunitary scattering matrix of $S = [0, 0; 1, 0]$, which can only be obtained if absorption losses are present. We also note that the superunitary version of an isolator ($S = [0, 0; 1, 1]$) requires active systems.

Interestingly, the generalization of passive isolators to three-port systems, known as circulators, corresponds instead to the unitary scattering matrix shown in the figure. Circulators transmit waves in a unirotational fashion, from port 1 to 3, 3 to 2 and 2 to 1, but never in reverse. They are capable of isolation, because one can use a circulator to build an isolator between, for instance, ports 1 and 2. In this case, the third port must be matched so that it acts as a perfect absorber, enabling passive two-port isolation at ports 1 and 2. These devices are often used in electronic systems to isolate sources from unwanted back reflection that originates at loads, or separate the emission and reception channels in full-duplex communication systems. They have been demonstrated in acoustical systems[15], which are relevant in many imaging or underwater communication systems.

In the literature, isolators have sometimes been referred to as 'diodes', despite the possible confusion with electronic diodes, which possess a distinct functionality. A diode is a nonlinear electronic component that lets a positive current flow in response to a positive voltage, but blocks current from flowing when subject to a negative voltage. This leads to signal rectification: sinusoidal voltage signals, with zero average, are transformed into positive-only current signals with a non-zero static (d.c.) component. Conversely, isolators are not meant to distort a sinusoidal incident field but allow power to flow only in one direction. In this Review, we only use the terms isolators and isolation, to avoid confusion with the concepts of diode and rectification.

The bias between positive and negative group velocities admits an interesting interpretation in terms of constant effective material parameters valid when the wavelength of the pump wave is small in comparison to the typical wavelength of the propagated waves. Hooke's law ceases to apply, as it systematically yields reciprocal behaviours, and is replaced by a constitutive law of the Willis type[52,53]; the Willis coupling coefficients in this case account for the nonreciprocal nature of the modulated microstructure. A constitutive law of the Willis form is characterized by an effective bulk modulus, κ^{eff}, an effective mass density, ρ^{eff} and a third parameter, S, which couples stress, σ, to velocity, $\vec{v}$, as well as momentum, $\vec{\mu}$, to strain, ε, namely, $\sigma = \kappa^{\mathrm{eff}} : \varepsilon + S \cdot \vec{v}$; $\vec{\mu} = -S : \varepsilon + \rho^{\mathrm{eff}} \cdot \vec{v}$. Note that, here, we have written the Willis constitutive relations in 'stress–momentum' form, but this is equivalent to the 'pressure–volume' strain form used in the example of BOX 1 (a one-to-one correspondence exists between the two formulations). While being linear and macroscopically time-invariant, this form of Willis coupling introduces a first-order time derivative to the motion equation, thus, breaking time-reversal symmetry and, consequently, reciprocity[47,51,54,55]. In the terminology of BOX 2, such Willis media are nonreciprocal because they break microscopic reversibility. Note that Willis coupling constants resulting from media with spatio-temporally modulated material properties are analogous to emergent nonreciprocal bianisotropic electromagnetic Tellegen media that result from modelling moving media in a static reference frame[56–58]. In this sense, this nonreciprocal Willis coupling has an influence similar to that of an externally applied magnetic field on the motion of charged particles. Note also that, whereas reciprocal Willis coupling has been experimentally observed[59], the experimental observation of nonreciprocal Willis coupling in a dynamic medium is particularly challenging, as it requires a fast, near-sonic, strong modulation of both elastic and inertial properties.

As the modulation speed tends to infinity, the material properties become solely time dependent and independent of spatial position. A structure with material properties varying periodically in time can display bandgaps in the wavenumber domain, that is, tilted by 90° in the frequency versus wavenumber band diagram. These vertical bandgaps produce wave motion with complex frequencies whose amplitude grows and/or decays everywhere in space exponentially in time. This amplification effect has been of interest for at least 60 years, first, for electrical transmission lines with time-varying capacitance[60]. Recent interest has turned to time-dependent acoustic[61] and elastic media[62], and space–time checkerboard patterns that lead to novel wave effects[63]. In practice, internal dissipation counteracts exponential parametric growth, with the result that the amplitude of the time-dependent modulation must exceed a critical value if it is to produce parametric growth[64].

Slow modulations and topological properties. Consider the scattering caused by slow pump waves, for which the phase shift $\dot{\gamma}\mathrm{d}t$ induced by the modulation over a small period of time $\mathrm{d}t$ is insufficient to trigger any transitions between dispersion branches. Accordingly, no significant

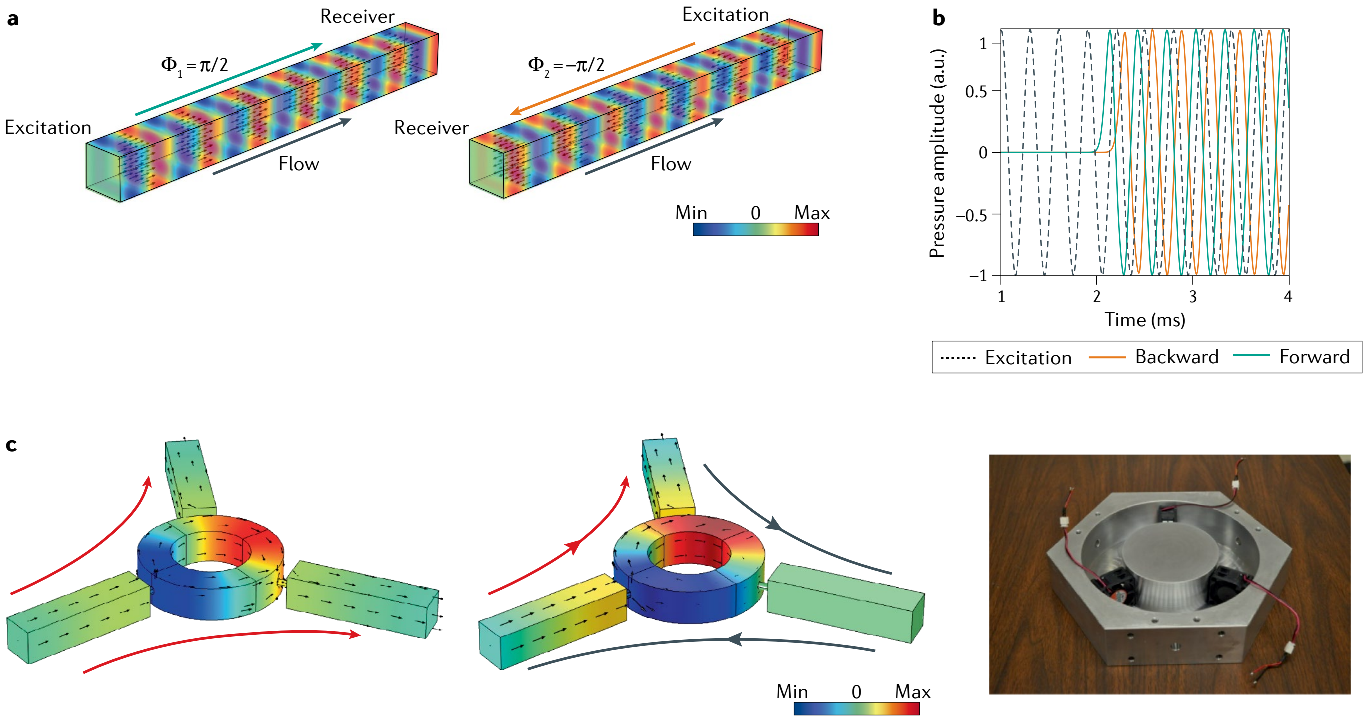


Fig. 1 | **Acoustic reciprocity from fluid motion. a** | In an acoustic pipe under constant air flow, sound propagates with different speeds, depending on whether it travels along or against the flow. Therefore, the transmission coefficients for forward and backward acoustic propagation can differ in phase. **b** | If the velocity of the flow and the length of the waveguide are adjusted to result in a 180° phase difference between forward and backward transmission, the system behaves as an acoustic gyrator[18]. **c** | A cylindrical cavity connected to three waveguides acts as a reciprocal power splitter at its dipolar resonance frequency (left). The system can be turned into a nonreciprocal circulator by rotating the air inside at a specific subsonic speed that depends on the total quality factor of the dipolar resonance (middle). Photograph of the first prototype of an acoustic circulator (top plate not shown), where fluid motion is realized using fans (right)[15]. Panels **a** and **b** adapted from REF.[18], CC BY 4.0. Panel **c** adapted with permission from REF.[15], AAAS. a.u., arbitary units.

nonreciprocal effects are expected to emerge over short observation periods. Remarkably, however, the phase shift $\gamma = \int_0^T \dot{\gamma}\,\mathrm{d}t$ accumulated over longer periods on the order of a modulation cycle $T = 1/f_M$ need not vanish and can cause significant overall changes in frequency and in propagation velocity. The phase increment $\dot{\gamma}$ is known as Berry connection, whereas γ is the Berry phase, a quantity originally introduced by Michael Berry[65] to explain nonclassical (quantal) interference phenomena induced by slowly changing environments, such as in the presence of externally applied magnetic fields and vector potentials[66] (topological properties are discussed in more detail in the next section). In the present context, Berry's phase, understood as slow but continual accumulation of Doppler shifts, can take opposite values for modes propagating in opposite directions. It, therefore, explains how and why the bands of a periodic medium get sheared and tilted under the influence of a modulation, albeit a slow one (FIG. 4a). As a matter of fact, each infinitesimal segment of a passing band shadowing an element $\mathrm{d}q$ of the Brillouin zone gets rotated by an amount equal to the Berry's curvature $\partial_q \dot{\gamma} / 2\pi$; the whole band tilts by an amount proportional to the total Berry's curvature $\int_{q\in \mathrm{BZ}} \int_0^T \partial_q \dot{\gamma}\,\mathrm{d}t\,\mathrm{d}q$ (REFS[47,48]). The total curvature is an example of a quantized topological invariant. Its value, an integer multiple of 2π, which is known as the Chern number, is immune to small perturbations affecting the pump wave and the background medium. Thus, a non-zero Chern number ensures the existence of a robust directional bandgap and constitutes one measure of nonreciprocity in a modulated medium.

Depending on applications, one-way mirroring can be regarded as a type of immunity against backscattering by defects[27]. That is because, in a directional gap, waves are constrained to propagate in a unique sense, thus, defects can only scatter forward (in the direction of the incoming wave). A pump wave, therefore, usually guarantees consistently high levels of transmission in specific directions (FIG. 4b). In some instances, however, the transmission is halted by a phenomenon of localization at boundaries. The localized vibration is then pumped up or down in frequency until it reaches a passing band, at which time, it can be sent back as a form of backscattering. In the position–frequency space, such vibrations manifest as edge modes circulating either clockwise or anticlockwise along the boundaries of the domain[37,48,67] (FIG. 4c). The difference between the number of clockwise and anticlockwise edge modes is also a topological invariant; in fact, by a principle of bulk–edge correspondence[68] that number is exactly equal to the Chern number. Consequently, a non-zero Chern number, in addition to controlling band tilting, indicates an imbalance between counter-propagating edge modes in clear violation of time-reversal symmetry.

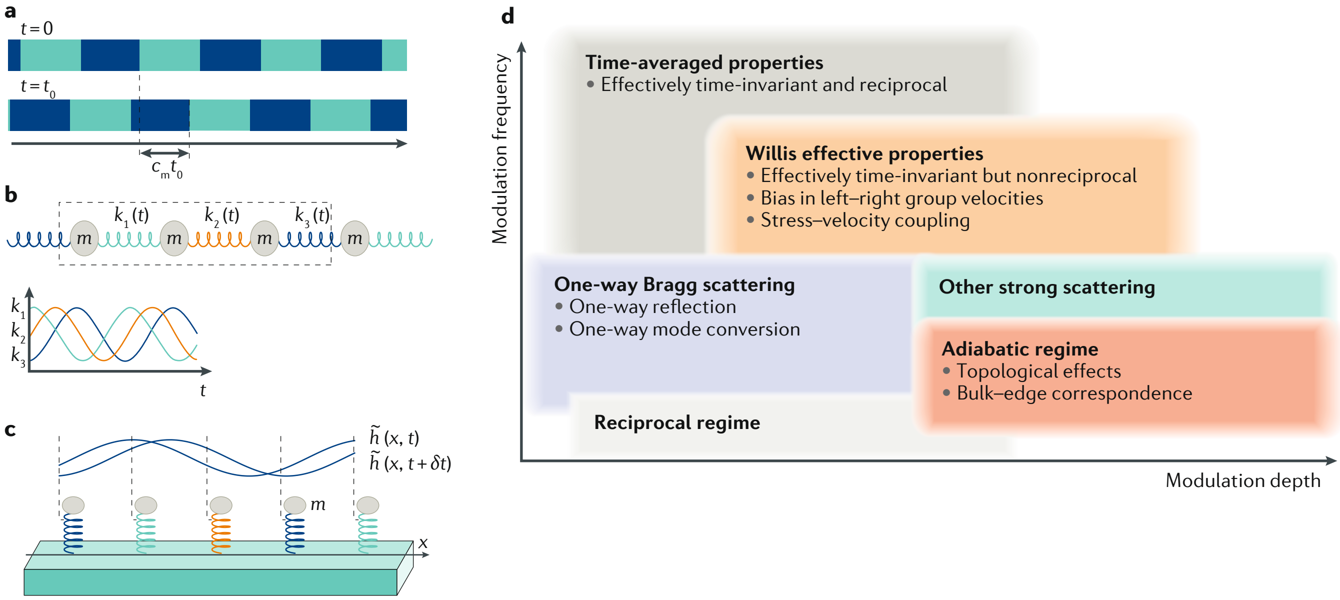


Fig. 2 | **Examples of activated media. a** | An activated two-phase laminate: interfaces are moving at the modulation velocity c_m (REF.[47]). **b** | An activated phononic crystal: bulk elastic moduli are periodically modulated in (*x*) space and (*t*) time, as indicated by the sinusoidal profiles below the spring–mass chain[48]. **c** | An activated metabeam: the elastic moduli are unchanged, but the stiffness (*k*) of the locally resonant attachments is modulated as a function of time and space[33]. **d** | Chart showing reciprocal and nonreciprocal regimes as a function of the modulation frequency and depth. Panel **a** adapted with permission from REF.[47], Elsevier. Panel **b** adapted with permission from REF.[48], APS. Panel **c** adapted with permission from REF.[33], Elsevier. *m*, mass.

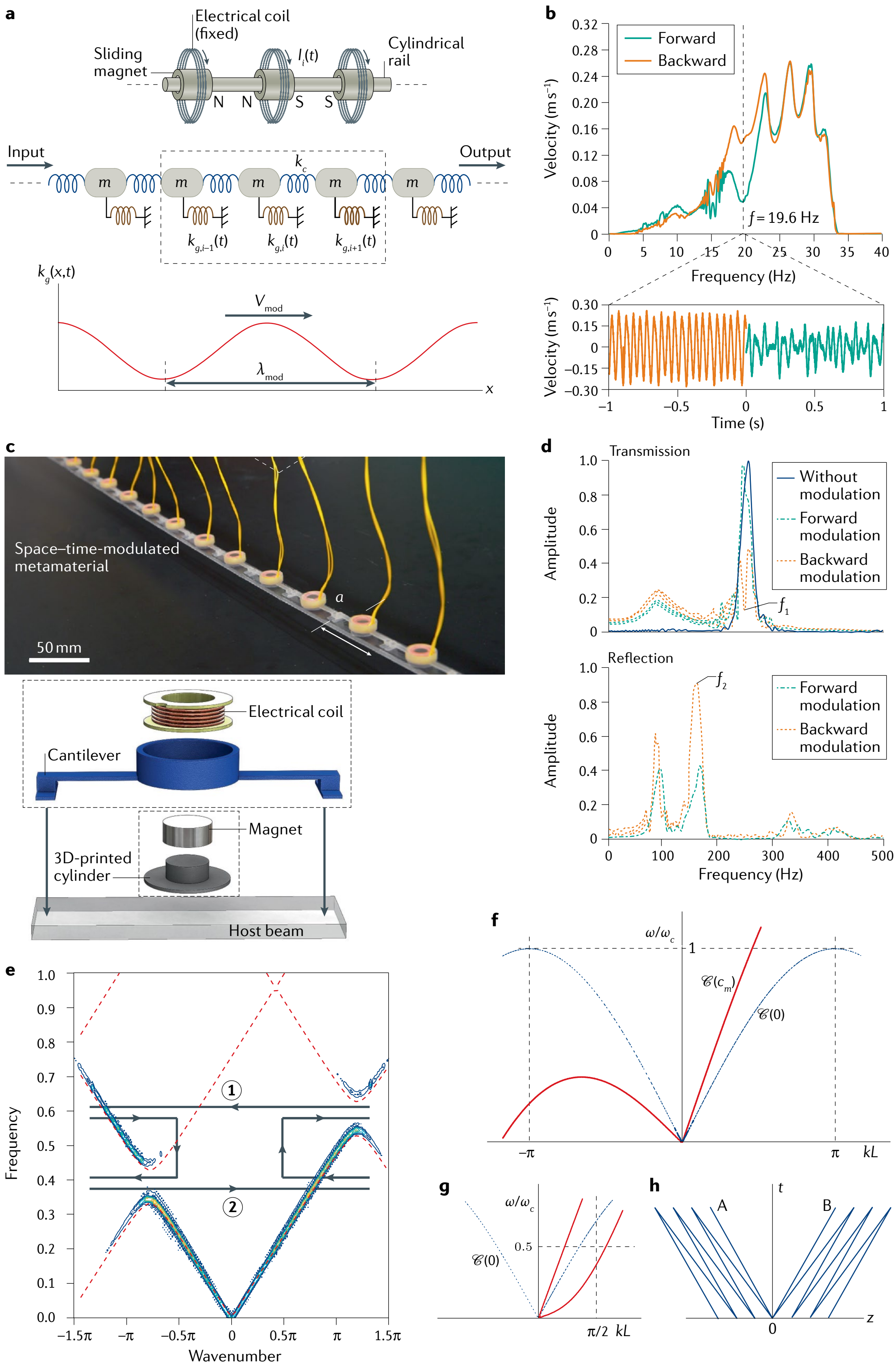
More generally, the Berry curvature and Chern number are straightforward to calculate for a given non-degenerate dispersion branch when the corresponding eigenmodes are functions of two parameters. Here, these two parameters are the 1D wavenumber and time. For 2D media, the two parameters can be the wavenumbers in the *x* and *y* directions. In that sense, two space dimensions are analogous to one space dimension plus time. The analogy permits to introduce similar concepts of topological invariants and bulk–edge correspondence, as well as potential connections with non-reciprocity, for 2D media, as discussed in the next section.

Nonreciprocity and topological edge states

Background. Wave motion supported by topologically protected edge states (TPESs)[69–72] can be extremely robust to backscattering and capable of nonreciprocal behaviour under certain circumstances. TPESs appear at the interface between two different insulators and are typically immune to scattering by defects. In this context, topology does not refer to the shape or geometry of the underlying medium but, for crystals, to the topological features of the dispersion of Bloch eigenmodes over the Brillouin zone. In contrast to other properties such as phase and group velocities or the gap width, a topological property is, by definition, insensitive to continuous perturbations, except those strong enough to close the bandgap[71]. In other words, the topology of an insulating crystal only changes when it is so perturbed that it stops being an insulator. Two insulators are then topologically inequivalent if they exhibit different topologies. At an interface between two inequivalent insulators, the topology changes and therefore, the system is forced to close its bandgap locally, which means that it must support localized edge states, which are the TPESs. TPESs display an inherent robustness to perturbations, including geometrical imperfections, structural disorder and impurities. This robustness is a result of the difference in the topology of the surrounding bulk insulators, independent of the interface details, which can only be destroyed by modifying the entirety of the bulk insulator, namely, closing their bandgaps, which requires unlikely global and large perturbations.

Of particular relevance to this Review are nonreciprocal TPESs that propagate along the interface in only one direction. Two classes of 2D topological insulators can be distinguished, depending on whether their edge modes are nonreciprocal or not: the Chern insulators, in which a non-trivial topology is created by breaking time-reversal symmetry and, as a by-product, support unidirectional nonreciprocal TPESs; and the spin Hall insulators, which rely on preserved time, and sometimes space, symmetries, and for which the TPESs come in time-reversed pairs (called spins) propagating in opposite directions, hence, preserving reciprocity. We stress that only the first kind of edge modes break reciprocity and is the principal focus of this section. However, the two time-reversed states of the second kind are decoupled; thus, they may be considered as featuring some form of one-way propagation for a certain class of loading restricted by the underlying preserved symmetry. For this reason, we also briefly mention works on reciprocal TPESs. The exceptional properties of both reciprocal and nonreciprocal TPESs have fuelled the interest of a wide community of researchers over the past few years, working on systems ranging from condensed-matter systems[73] to photonic[70], acoustic[72], elastodynamic and mechanical systems[69].

Starting from the work in condensed-matter physics of Duncan Haldane[73], who predicted the possibility of unidirectional electronic edge states, TPESs in quantum systems have been the subject of intense research owing to their potential in a number of technological areas. Nonreciprocal electronic propagation[74], for example,



is based on breaking time-reversal symmetry to produce one-way chiral edge modes between bulk bands that are characterized in terms of their Chern number as a topological invariant. More recently, Charles Kane and Eugene Mele[75,76] discovered reciprocal topological modes in systems with intrinsic spin orbit coupling that exhibit the quantum spin Hall effect (QSHE). These systems, whose unique behaviour was demonstrated experimentally[77], do not require breaking of time-reversal symmetry and are, therefore, of particular interest as blueprints to establish analogies in other physical domains. In fact, these quantum-mechanical phenomena have been pursued in diverse areas of physics as in the case of the work of Haldane and Srinivas Raghu[78], who demonstrated boundary modes in electromagnetic systems following Maxwell's relations. They have been subsequently investigated in acoustics[79–81], photonics[71,82] and mechanics[83–87], as well as in coupled-wave domains, such as optomechanics[88]. In all of these different domains, properties such as lossless propagation, existence of waves confined to a boundary or to an interface, immunity to backscattering and localization in the presence of defects and imperfections are all related to band topology. Although the existence of TPESs does not necessarily imply nonreciprocal wave motion, reciprocity breaking is one way to induce topological order in acoustics and mechanics. In addition, geometrical symmetry breaking, as in the case of chiral symmetries, while not breaking reciprocity, can be used to support topologically robust one-way wave motion that does not backscatter in the presence of disorder, provided that this disorder does not couple opposite chiral states. In this section, we discuss nonreciprocal topological insulators, and we briefly review reciprocal systems supporting edge-bound propagation of sound and elastic waves to provide a broader context.

◀ Fig. 3 | **Experimental demonstrations of nonreciprocity in dynamic media.** **a** | A magnetically activated phononic lattice, with its equivalent grounded spring–mass chain system: adjacent ring magnets of mass m repel each other with an equivalent stiffness k_c; out-of-phase alternating currents I flowing through the coils generate an equivalent grounding stiffness k_g with wavelength λ_{mod} and phase velocity V_{mod}. **b** | Measured end-to-end frequency-response function for the system in panel **a**. The forward (left-to-right) and backward (right-to-left) response functions are nonreciprocal near 20 Hz. The measured time series at the last moving mass shows that the forward configuration (shown from 0 to 1 s) has a different temporal character to the backward configuration (shown from 0 to −1 s). **c** | A magnetically activated metabeam[31]: resonators in the form of electrical coils supported by cantilevers oscillate coaxially around magnets bonded to the host beam; the coils' out-of-phase electrical input generates a space–time pump wave. **d** | The measured transmission and reflection spectra for the system in panel **c**: frequency f_1 is transmitted when the incident wave travels with the pump wave, but is reflected into f_2 when it travels against it. **e** | A typical dispersion diagram of a weakly modulated medium featuring a pair of one-way bandgaps (colour map: numerical simulation; dashed line: analytical approximation)[28]. One-way bandgaps explain how a nonreciprocal one-way mirror operates; frequencies about bandgap (1) are transmitted if incident in one direction (here, right-to-left), but are reflected into frequencies of bandgap (2) if incident in the opposite direction (here, left-to-right); the arrows illustrate these transmission/reflection transitions. **f** | Example of bias in group velocities of a strongly modulated medium[47]: $\mathscr{C}(0)$ is the non-modulated reference, $\mathscr{C}(c_m)$ is modulated at velocity $c_m = c/10$, where c is the phase speed of propagating acoustic waves in the non-modulated medium when $kL \ll 1$. **g** | Reversal of group velocities under a high-frequency strong modulation[47]: group velocities $c_\pm$ are of the same sign; here, $c_m = 2c/3$. **h** | Example of characteristic lines $z - c_\pm t = cst$ in a modulated medium with reversed group velocities: $c_\pm > 0$ for $z > 0$ and vice versa, so that no signal can reach $z = 0$ (REF.[51]). Panels **a** and **b** adapted with permission from REF.[30], APS. Panels **c** and **d** adapted with permission from REF.[31], APS. Panel **e** adapted from REF.[28], CC BY 3.0. Panels **f** and **g** adapted with permission from REF.[47], Elsevier. Panel **h** adapted by permission from Springer International Publishing: Springer *An Introduction to the Mathematical Theory of Dynamic Materials*, by Lurie, A. K. ©2017 (REF.[51]).

QHE analogues. There are two broad ways to realize topologically protected wave propagation in acoustic and elastic media. The first one seeks to mimic the quantum Hall effect (QHE) by breaking reciprocity through active components. Changing the parity of active devices or modulating the physical properties in time, for example, has been shown to alter the direction and nature of edge waves[27,47]. Examples include the use of magnetic fields in biological systems[89], lattices of gyroscopes[90,91] (FIG. 5a) and acoustic circulators operating on the basis of a flow-induced bias[80,92–97]. In particular, edge modes were demonstrated in biological structures in which time-reversal symmetry is broken by Lorentz forces on ions produced by weak magnetic fields[89]. A systematic way to analyse eigenvalue problems that break time-reversal symmetry by gyroscopic forces was also developed[98], and a lattice of gyroscopes was experimentally realized[90] (FIG. 5a). Theoretical studies have investigated the existence of chiral edge modes in hexagonal and square gyroscopic lattices[99] (FIG. 5b). Other proposals involved moving fluid to break time-reversal symmetry at the inclusion level to induce topological order[92,93] (FIG. 5c,d), a concept that has been demonstrated experimentally[95], which is consistent with the use of Coriolis forces to manipulate Dirac cones by spinning the entire lattice[17]. Several systems supporting TPESs that rely on the modulation of the strength of the interactions (stiffness) in time have been proposed[27,67,80,96,100]. In general, this body of works pursues methods for breaking time-reversal symmetry through the introduction of an external bias, which modulates the strength of interactions or the inertial properties of the system and provides the ability to produce chiral TPESs. As such, these states are associated with wave motion that is strictly nonreciprocal. For these reasons, QHE analogues can be considered as examples of media referred to in this manuscript as activated materials.

QSHE and QVHE analogues. The need for an external bias, and for the associated external source of power, has motivated the search for QSHE analogues that employ only passive components. This is particularly compelling for mechanical metamaterials, for which the need for an external bias to break reciprocity may pose a substantial challenge to the practical application of one-way wave motion. For this reason, material platforms that are fully reciprocal and that feature both forward and backward propagating helical edge states have attracted substantial interest. Owing to geometrical asymmetries, two oppositely propagating modes support opposite chirality, and can propagate with robust properties protected by topology, as long as disorder, truncations or lattice defects do not couple the two helicities. This is generally guaranteed as long as the entire system, including the defects, respects a given symmetry or an accidental spin degeneracy, in addition to time-reversal symmetry itself. Examples of QSHE analogues are illustrated in several studies by way of both numerical[83,84,101,102] and

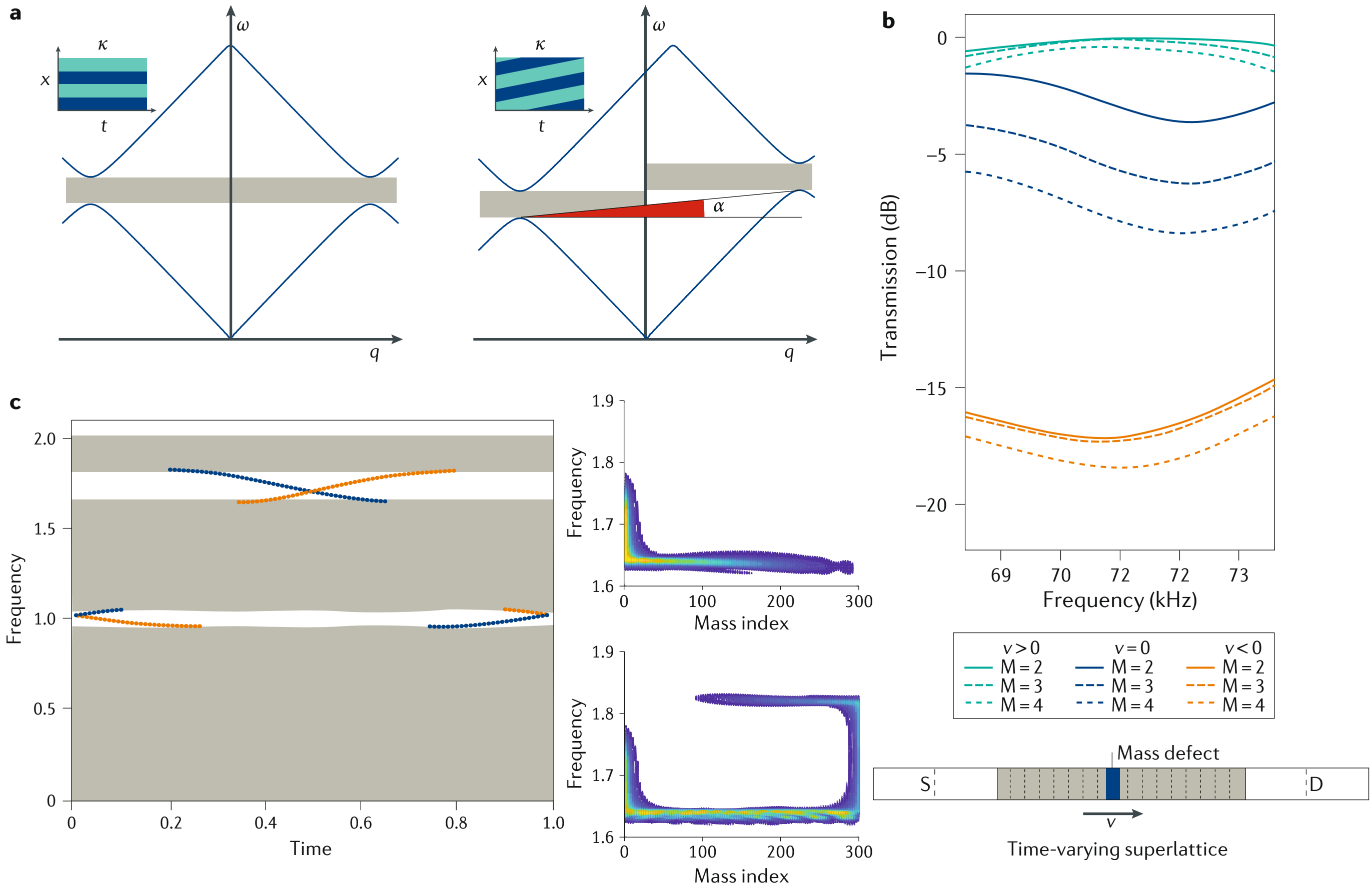


Fig. 4 | **Topological effects in dynamic media. a** | The modulation of the constitutive parameters, such as the bulk modulus *κ*, results in a tilting of the dispersion band by an angle *α*. The direction-independent bandgap is transformed into two one-way bandgaps[48]. **b** | Robust transmission in dynamic media. The schematic shows a modulated system supporting unidirectional propagation including a mass defect. The green and orange curves correspond to opposite modulation velocities (*v*); the blue curve corresponds to a time-independent system. For positive modulation velocity, the presence of the defect does not hinder transmission, thanks to the absence of backscattering[27]. **c** | Topological one-way edge states as predicted by bulk–edge correspondence. The eigenfrequencies of the edge states are shown as a function of time (normalized to one modulation period), and they cross both bandgaps. Blue and orange indicate the left and right edge modes, respectively. The two *n*–*ω* subplots show two snapshots of the adiabatic evolution of the edge mode within the second gap; *n* is the mass index in a finite lattice[67]. Panel **a** adapted with permission from REF.[48], APS. Panel **b** adapted with permission from Swinteck, N. et al. *J. Appl. Phys.* **118**, 063103 (2015)[27]. Panel **c** adapted with permission from REF.[67], APS. M, mass.

experimental[87,103,104] investigations, which involve coupled pendulums[103], plates with two scale holes[83] and resonators[84,87], as well as electrical circuits[104]. Among these, one of the first experimental demonstrations of edge states was obtained[103] in a square lattice network of double pendulums connected by a network of springs and levers. Also, helical edge modes in perforated thin plates were numerically demonstrated[83] by coupling symmetric and antisymmetric Lamb wave modes. Similar concepts were exploited for the realization of topological waveguides based on resonant spirals cut out from thin plates[105]. The results in REF.[83] have also provided guidelines for the experimental demonstration of a QSHE analogue in the form of a continuous elastic plate[106]. In another example, a mechanical analogue of the in-plane QSHE was realized in a hexagonal kagome lattice using the Brillouin zone folding technique[102]. Chiral symmetries can also be used to induce topological order in acoustic systems[97,107], a concept also extendable to higher-order topological phases, such as robust corner states[108].

A parallel line of work, again resulting in reciprocal systems, employs valley degrees of freedom (which are natural time-reversed modes belonging to band extrema with opposite Bloch wavenumbers) and the topological differences generated by symmetry inversions within a unit cell to achieve TPESs in structures that emulate the quantum valley Hall effect (QVHE). Notably, the QVHE exploits valley states, which have the same frequency but different Bloch wavenumber, instead of spin states, which are perfectly degenerate time-reversed states. The advantage is that each lattice site needs to have only one degree of freedom, which provides the opportunity to obtain configurations of reduced geometrical complexity. Valley degrees of freedom arise naturally in systems with time-reversal symmetry and have been predicted theoretically in graphene[109,110], where wavefunctions at opposite valleys feature opposite polarizations and, thus, emulate spin–orbit interactions. This concept was extended[111] to a photonic crystal exhibiting topologically protected valley edge states, and valley modes have been predicted[112]

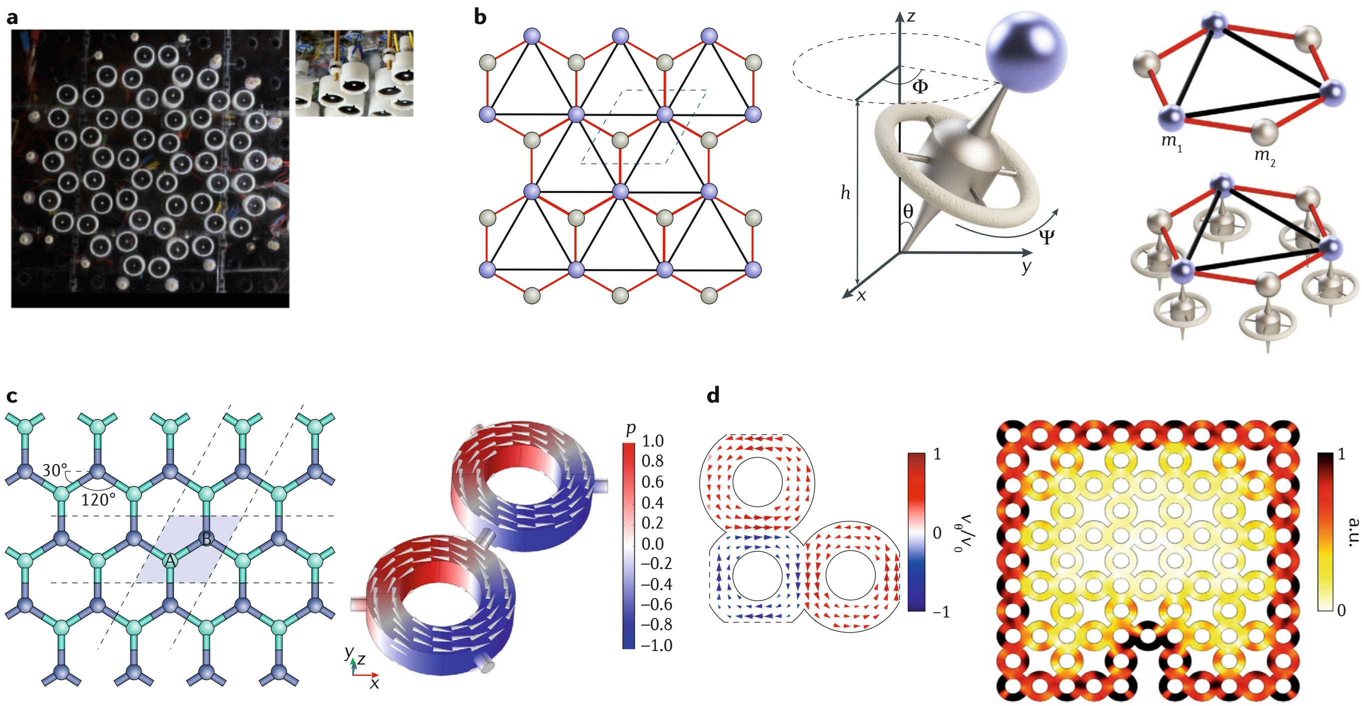


Fig. 5 | **Quantum Hall effect analogues in mechanics and acoustics. a** | Experimental implementation of a topological lattice of coupled gyroscopes from REF.[90]. **b** | Schematic of the hexagonal gyroscopic lattice investigated in REF.[99]. Blue and grey spheres have the same mass, and the red and black lines represent springs with different stiffness constants. A schematic of a gyroscope with the tip pinned to a mass in the lattice is shown, together with a schematic of the unit cell for the gyroscopic phononic crystal. **c** | First proposal of lattices of acoustic circulators as Chern insulators. One unit cell of the lattice is shown, with the acoustic pressure (*p*) distribution shown in colour for one of the Dirac modes. The arrows indicate the direction of air flow[92]. **d** | A Chern sonic insulator in a fluid network with a self-organized flow of self-propelled particles. The schematic shows the steady state of the polar active liquid in a confined geometry based on the Lieb lattice; blue and red indicate opposite directions of movement. The image on the right shows an eigenmode whose frequency is in the bandgap; the edge mode is clearly seen and is robust against a defect at the lower edge[93]. Panel **a** adapted with permission from REF.[90], PNAS. Panel **b** reprinted with permission from REF.[99], APS. Panel **c** adapted from REF.[92], CC BY 4.0. Panel **d** adapted from REF.[93], Springer Nature Limited. a.u., arbitary units.

in photonic crystals with a hexagonal lattice of inclusions. More recently, this approach has been extended to acoustic waves propagating in a 1D phononic crystal, where the Zak phase is the topological invariant employed to assess the non-trivial nature of the bulk gaps[113]. A subsequent extension to 2D acoustic domains employed triangular stubs to break inversion symmetry by varying their orientation with respect to the lattice[114–116]. Initial studies in mechanical metamaterials include numerical work[117] and the experimental demonstration of a QVHE analogue in the form of a hexagonal lattice with lumped masses inserted at locations that break the C_{3v} symmetry inherent to the hexagonal geometry, while preserving the C_3 symmetry[118]. A similar approach was implemented in REF.[119], while the idea of enlarging a unit cell to induce zone folding at the Γ point in reciprocal space, which leads to a double degeneracy, was proposed for nanoscale mechanical systems[79] and realized through an array of resonators[120]. Also, topological Stoneley waves propagated along the edges of kagome lattices were investigated[121] by using an asymptotic homogenization technique that transforms the discrete motion equation of the lattice into a continuum partial differential equation. This study was the first adaption of the QVHE to the in-plane motion of mechanical lattices. As explained above, the edge modes of a quantum valley Hall insulator are strictly reciprocal, but apparent nonreciprocity can be observed for defects that do not couple valleys together, in particular, valley-preserving interface turns[122].

Nonreciprocity in nonlinear media

In elastic and acoustic media in which the effective material properties do not change with time (passive media), the absence of reciprocity is, perhaps, most commonly associated with material nonlinearity. Although reciprocity does not generally hold in nonlinear materials, these media do not always (or even necessarily) behave in a nonreciprocal fashion. Spatial asymmetry is a necessary ingredient for breaking reciprocity and, even then, nonreciprocity is not guaranteed for all parameter ranges. We focus the discussion on the familiar point-to-point form of reciprocity as it applies — or not — to passive, nonlinear, non-gyroscopic systems with time-invariant properties and boundary conditions. Given that the vast majority of the literature in this topic concerns 1D waveguides, we emphasize systems that can be adequately modelled using a combination of scalar wave fields and coupled oscillators.

The salient feature of nonlinear media is that their dynamic behaviour depends on the amplitude of motion.

To operate, nonlinear, nonreciprocal devices rely on this dependence, which, itself, is a function of various parameters and can lead to different nonlinear phenomena (FIG. 6a). The most common mechanisms responsible for nonreciprocal dynamics are generation of higher harmonics, nonlinear resonances and bifurcation (a sudden change in the nature of the response due to a small change in a system parameter)[123,124]. Operating at relatively low amplitudes of motion, where the frequency of the nonlinear response is mostly preserved, may also lead to nonreciprocity both at finite frequencies[125,126] and at zero frequency (static nonreciprocity[127,128]) (FIG. 6b). We note that other processes may also be utilized for violating reciprocity, such as solitary waves[129] and acoustic radiation pressure[130]. We note also the possibility to trigger highly nonlinear responses at low amplitudes using feedback-controlled electro-acoustic elements[21,22]. Whether static or dynamic, two ingredients are necessary for circumventing the constraint of reciprocity: nonlinearity and spatial asymmetry.

Before diving into a more systematic description of these approaches, it is worth stressing that nonlinearity-based nonreciprocal systems hold a few fundamental limitations compared with the externally biased systems considered in the previous sections. First, a passive nonreciprocal device can support drastically different transmissions for oppositely propagating waves, but it cannot ensure isolation when the system is excited simultaneously from both sides[131]. In this sense, these devices cannot operate as conventional isolators to protect a source from back reflections, because these reflections can trickle through the isolated port in the presence of an outgoing signal. This is a result of the fact that the superposition principle does not apply to nonlinear systems. In addition, there is a trade-off between the degree of nonreciprocity achievable in passive, nonlinear resonators and the magnitude of forward transmission[132,133].

In the following, we provide an overview of the different strategies available to realize nonreciprocity based

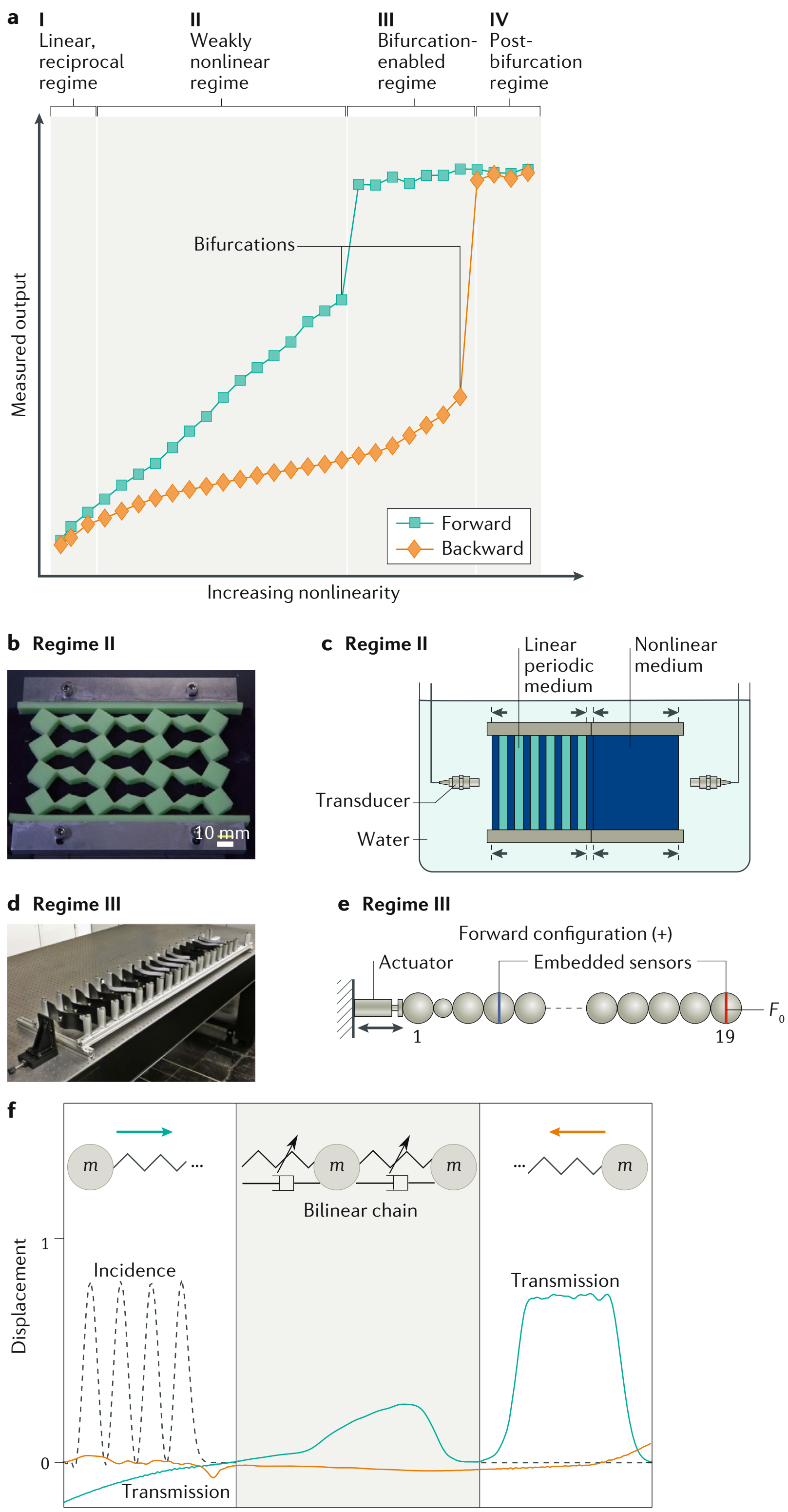


Fig. 6 | **Overview of nonreciprocity in nonlinear media.** **a** | The degree of nonlinearity increases along the horizontal axis; for example, owing to an increase in the input energy or forcing amplitude. The monitored output of the nonlinear medium is plotted on the vertical axis; for example, the amplitude of motion. Generically, four different response regimes exist for amplitude-dependent nonlinear systems. Regimes II and III are useful for the design of nonreciprocal devices and materials, with examples provided in panels **b**–**e**. **b** | Static nonreciprocity is observed in the large-deformation response of an asymmetric structure. **c** | A layer of a nonlinear medium is attached to a linear periodic waveguide (superlattice) to create a system that can break reciprocity via second-harmonic generation. **d** | Long-range unidirectional wave transmission is observed in a periodic arrangement of identical bistable structural elements. Each element comprises a magnetic mass at the centre of a pre-buckled sheet. To break reciprocity, asymmetry is incorporated within the potential energy of each element. **e** | A defective unit (a smaller bead) is embedded within a uniform granular chain to break the spatial symmetry and enable bifurcation-based nonreciprocity. **f** | An example of an amplitude-independent nonlinear medium is a chain of springs and masses containing a finite region of bilinear springs with constants that depend only on the sign of the displacement or strain. Panel **b** adapted from REF.[127], Springer Nature Limited. Panel **c** adapted from REF.[141], Springer Nature Limited. Panel **d** adapted with permission from REF.[174], ASME. Panel **e** adapted from REF.[161], Springer Nature Limited. Panel **f** adapted with permission from REF.[163], ASME.

on nonlinearities. We focus on a phenomenological description and comparison of these strategies, rather than on a detailed quantitative discussion.

Weakly nonlinear media. It is helpful to begin with examples of reciprocal nonlinear response. The most straightforward scenario is that involving a spatially symmetric system, for which exchanging the source and receiver has no effect on signal propagation. Any form of spatial symmetry guarantees reciprocal propagation of waves, regardless of the type and degree of nonlinearity. An example would be a granular chain of identical, spherical beads with the input and output located at the two ends[134].

Mirror symmetry of a reciprocal nonlinear medium may be broken in various ways. Examples include gradually changing the properties of the medium[135–137], using asymmetric unit cells in periodic waveguides[138], incorporating nonlinear defects or interfaces[139–148], breaking the symmetry of the functional form of the nonlinear internal forces[126,128,149–151] or simply having the input and output points at asymmetric locations[152]. An asymptotic analysis may be adopted to demonstrate these results in a general form for a weakly nonlinear system subject to an impulsive load[152]. It can be shown, in this context, that even the boundary conditions can play a role in determining whether the response in a specific configuration is reciprocal.

A nonlinear medium can generate multiple harmonics of a wave travelling through it. The amplitudes of the extra harmonics, and whether they are superharmonics or subharmonics, depend on the nature of the nonlinearity and the frequency of the incident wave. At low energies (weak nonlinearity), the dominant frequency is typically the second harmonic, but it is the third harmonic if the nonlinear force is odd symmetric (as in the case of a cubic restoring force). Reciprocity is broken if the transmission of (some of) the harmonics are altered upon interchanging the source and receiver locations. This principle was used to operate an acoustic isolator[141,153]. The researchers attached a layer of a nonlinear medium to a linear periodic waveguide (FIG. 6c) and fixed the frequency of the incident wave within the bandgap of the linear medium such that its second harmonic would fall within a pass band. By choosing the amplitude of the incident wave properly, they created a system in which, if the wave comes from the nonlinear side, its second harmonic is generated and passes through the linear medium to the other side; if the wave comes from the linear side, it is filtered out before reaching the nonlinear medium and negligible energy is transmitted to the other side.

Another nonlinear phenomenon that can be used for nonreciprocal transmission is the energy dependence of resonance frequencies. For instance, consider wave transmission through a linear layered medium that contains an asymmetric nonlinear portion[123]. It is relatively straightforward to set up the layers such that the wave amplitudes are different on the two sides of the nonlinear layer. For the near-resonance dynamics, the combination of asymmetry and nonlinearity leads to different transmissions through the nonlinear layer when the source and receiver are interchanged. The appeal of this scenario is that the frequency of the input is mostly preserved. This phenomenon was experimentally demonstrated using granular chains[125].

Whether the realization of nonreciprocity is based on the generation of extra (sub or super) harmonics or the shifts of resonances, the underlying principle is the same. We can regard both cases as 1:n nonlinear resonances, where n denotes the dominant frequency of the response dynamics. We have $n=2$ for second-harmonic generation, for example, and $n=1$ when relying on shifts occurring at the same frequency as the external driving. These principles work for both transient and steady-state dynamics.

Bifurcation-based nonreciprocity. Bifurcations are the points of departure from a weakly nonlinear behaviour, characterized by a qualitative change in the nature of the response in the phase space[154,155]. Different types of bifurcations can occur depending on the nature of the response (such as static equilibrium or periodic motion).

One familiar bifurcation in mechanics is the bistable behaviour of elastic members in compression, for example, the buckling of a column. Bistability within the unit cell of a lattice material was used to demonstrate long-range unidirectional transmission of waves[156,157]. The researchers incorporated asymmetry in the potential energy of each unit cell, such that different energies are required for transitioning from one energy well to the other (FIG. 6d). Provided that the initial configuration of the lattice is chosen appropriately, it becomes possible to transfer energy through the entire lattice in the direction of decreasing potential energy. This unidirectional transmission occurs with minimal loss and dispersion (change in the waveform) because the underlying mechanism is a change in the static equilibrium of the unit cells.

A similar bistability can happen in a medium in periodic motion, for example, in response to an external harmonic excitation. For instance, consider a 1D layered medium consisting of nonlinear unit cells, with an asymmetric potential-energy function. The system can be assembled such that, when the excitation source is placed in different locations (for example, at either end of the medium), the information travels through different energy landscapes. Accordingly, as the driving amplitude increases, the response corresponding to one of the two source locations loses stability first[124]. As a result, the two outputs become markedly different. In this scenario, the response is understandably nonreciprocal, even before reaching the bifurcation point. If the bifurcation results in a significant change in the response, particularly in terms of energy transmission, then the post-bifurcation nonreciprocity is more pronounced. To achieve this, one strategy is to use the supratransmission phenomenon[158–160], in which bifurcation results in loss of time periodicity.

Consider a nonlinear periodic medium that is subject to harmonic excitations, with a frequency within the linear bandgap. No energy is transmitted through the medium at low driving amplitudes, owing to the dispersive property of the periodic medium. Beyond a critical driving amplitude, however, the low-energy harmonic motion loses stability and a nonlinear wave with a broadband frequency content is generated. Thus, the amount of transmitted energy increases by orders of magnitude, and its spectral contents lie within the linear pass band of the medium (again, owing to dispersion). If the nonlinear medium is not symmetric, then the onset of supratransmission may depend on the location of the source. This principle was used to develop a rectifier for elastic waves by embedding a defective unit within a uniform granular chain to break the symmetry of the system[161] (FIG. 6e). This enabled a very high contrast (four orders of magnitude) between the transmitted energies when the source and receiver were interchanged.

Relying on bifurcations for achieving nonreciprocity is naturally more suitable when the device is operated near the bifurcation point. The performance of bifurcation-based nonreciprocity depends on the driving amplitude, among other parameters. For example, there is a value of driving amplitude above which bifurcations occur for both the forward and backward configurations. Although the response might still remain nonreciprocal in this case, the nonreciprocity is less marked. Additionally, the post-bifurcation response is often quasi-periodic or chaotic[124,159,160], making it cumbersome to compare two such outputs in a reciprocity test. Even when the post-bifurcation response remains time-periodic (for example, owing to a period doubling[126]), nonreciprocity is clearly more pronounced in the parameter range where bifurcation occurs for only one of the forward and backward configurations.

The post-bifurcation behaviour of the response, as well as the instability mechanism, depends on various properties of the system, such as the type of nonlinearity, energy loss, structural imperfections, finite size effects and operating frequency. For supratransmission, these effects are discussed in REFS[160,162].

Nonreciprocity in bilinear media. Bilinearity is a special case of elastic nonlinearity characterized by a stiffness that transitions between two linear states at a critical load. When this transition occurs at zero loading, the effective stiffness depends on whether the local 1D deformation is in a state of compression or extension. In this case, unlike other nonlinear stiffness models, the bilinear relation is amplitude independent in the sense that, even for very small amplitudes of oscillations, the constitutive law is nonlinear: the nonlinearity enters only through the sign of the displacement. Bilinear, also known as bimodular, material response has been proposed as a model for studying contact and for elastic solids containing cracks. Bilinear media can exhibit unique families of complex dynamic behaviour, owing to the non-smooth nature of their constitutive material law[155]. Bilinearity may be regarded as the simplest departure from linearity that can produce nonreciprocal static and dynamic behaviour.

A single bilinear spring is spatially symmetric and is, therefore, a reciprocal system. Nonreciprocity can be obtained in two-degrees-of-freedom spring–mass systems, with asymmetric, bilinear spring arrangements[128]. Bifurcation-based nonreciprocity may be realized in lattice materials with bilinear elasticity[126]. Amplitude-independent nonreciprocal systems may be designed by combining multiple bilinear springs in 1D spring–mass chains[163]. In this model, the indispensable ingredient of asymmetry is achieved using spatially varying bilinear spring properties (FIG. 6f). A recent experimental study considered nonreciprocal propagation in a linear spring–mass system in which only one of the springs was bilinear[151].

FIGURE 6 summarizes the different scenarios discussed for nonreciprocity in nonlinear media. Whether nonreciprocity is based on weakly nonlinear behaviour, bifurcation or sign-dependent bilinearity, we emphasize that simultaneous presence of nonlinearity and spatial asymmetry are necessary, but not sufficient, conditions for the existence of nonreciprocal dynamics.

Nonlinear dynamic media. We have been careful so far in our discussion to avoid any overlap between passive nonlinear media and dynamic linear media. Unfortunately, it is not possible to make a general claim on whether nonlinearity intensifies the nonreciprocal bias of a modulated medium or pacifies it. Like any other nonlinear problem, the answer depends on a variety of different parameters, such as the type and strength of nonlinearity, the spatial and temporal nature of the excitations, the frequency, wavenumber and amplitude of modulations, and the energy loss within the system. Conclusions can be made only on a specific, case-by-case analysis.

One scenario is that nonlinear forces monotonically intensify the nonreciprocal bias in a time-modulated medium. This is very similar to how nonlinearity affects nonreciprocity in a passive medium, with the main difference being that the linear regime would no longer be reciprocal in a modulated medium (regime I in FIG. 6). The possibility of the existence of this scenario was verified numerically[164] for the set-up presented in REF.[30], with the amplitude of modulations acting as the control parameter for nonlinearity (the horizontal axis in FIG. 6).

Conclusions and perspectives

As a principle of nature, reciprocity is remarkably tenacious. It persists in the presence of heterogeneity on any length scale (spatial asymmetry) and internal damping (time irreversibility). Reciprocal wave motion is guaranteed if the medium is time-invariant, passive and linear with microscopic time reversibility, conditions that are realized for most materials in daily life. For these reasons, one can conclude that reciprocity is a robust fundamental principle that is hard to beat.

In this Review, we have presented many specific approaches that have been proposed and employed to purposefully violate one or more of the requirements for reciprocal wave motion in acoustic and elastic materials. The most familiar condition for breaking reciprocity is the presence of mean flow. In this scenario, flow breaks the microscopic time reversibility, leading to nonreciprocal effects, resulting in a system analogous to a gyrator when the flow is parallel to wave motion, or a circulator otherwise (BOX 3 and FIG. 3). The condition of time-invariance is broken in activated media, in which material properties are varied in both space and time. These materials can display strong nonreciprocal effects if the space–time modulation is large enough (FIG. 2). Various mechanisms are available to realize the level of active control necessary to modulate material properties, for example, those realized in piezoelectric materials and magnetic elements (FIG. 3), which make activated materials a practical and promising candidate for applications in elastic wave control seeking to make use of nonreciprocal signal transmission. The dynamics of space–time-modulated materials can be explained in terms of new descriptors of physical states, such as the Willis dynamic effective medium equations and by topological invariants in the frequency–wavenumber domain (FIG. 4). Aside from activated materials, there is a deeper connection between one-way wave propagation and the topological description of quasi-periodic systems. Although this has analogies in electronic systems, the mechanical realizations are quite distinct (FIG. 5) and can exhibit robust one-way propagation effects. Finally, material nonlinearity combined with spatial asymmetry provides a distinct route to nonreciprocity (FIG. 6). Nonlinearity remains an important tool for realizing nonreciprocity in elastic and acoustic materials.

The optimal utilization of nonlinearity for operation of nonreciprocal devices is an active area of development, despite the intrinsic limitations of passive, nonlinear materials in comparison to linear, externally biased materials. Even more remarkable outcomes are expected when nonlinearity is utilized in combination with another reciprocity-breaking mechanism. For example, recent studies have focused on the nonlinear dynamics of topologically protected edge states[165,166]. The combined effects of nonlinearity and spatio-temporal modulations have not yet received much attention in the literature. In the light of recent advances in modelling complex nonlinear and spatio-temporally varying elastic materials[46,167] and experimental realizations of dynamic elastic media[30,31,42,168,169], we expect the study of nonreciprocity to extend to modulated nonlinear media in the near future. Beyond the innate appeal of the underlying physical phenomena, the interest in the dynamics of these media is partly motivated by the fact that nonlinear forces are inevitably present in experiments. Another new perspective was recently presented by Denis Bartolo and David Carpentier[170] in connection with the elasticity of surfaces with non-orientable topology, such as a Möbius strip. Despite having a linear constitutive law, the deformation of a Möbius strip subject to shear stress is intrinsically nonlinear and the elastic response is not reciprocal. This counter-intuitive property of non-orientable surfaces can potentially lead to yet another class of nonreciprocal elastic materials.

All of the techniques to elude reciprocity have their own benefits and drawbacks. Active materials, which include systems with mean flow as well as spatio-temporal modulation of effective properties, require constant sources of energy. Conversely, nonlinear materials are passive systems requiring no external impetus. However, they use the input power of the source, which may come with other drawbacks and limitations on the isolation of simultaneous sources, isolation level or bandwidth, as previously noted in photonics[131,171]. Nonlinear materials also rely on more complicated constitutive relations that are not easily designed or created using existing fabrication technologies. Topological wave systems are inherently narrowband in frequency and wavenumber space and confined to the boundaries of engineered-material domains. Yet, individual systems can take advantage of these hurdles in a constructive manner to effectuate nonreciprocal wave motion. The task before the materials designer is to consolidate all possible effects in order to generate a nonreciprocal response for a given application. Owing to the complexities listed above, many of the proposed examples found in the literature to demonstrate nonreciprocal acoustic behaviour may not be easily realizable, with many requiring advances in fabrication capabilities. Further, we anticipate that future approaches to violate reciprocity may combine mechanisms discussed in this Review or propose entirely new mechanisms or approaches that have not yet been considered. Some possible avenues include the use of transmission-frequency windows via coupling of bandgaps with nonlinear frequency shifting. A particularly interesting possibility[172] combines acoustic waves under the small-on-large effects of elastic prestress with nonlinearity in two distinctly different modes: dynamically to detune the signal frequency and statically to change the effective elastic stiffness. Other possible routes combine bilinearity as the frequency converter with a linear wave filter[151].

This Review is focused on the direct problem of how to design materials with embedded nonreciprocal mechanisms to add an additional degree of control to propagating waves. The inverse problems posed by nonreciprocity are also relevant and could lead to new techniques for assessing material properties. Just as one can “hear the shape of a drum” (REF.[173]), if one knows the modal frequencies, it may be possible to infer nonlinear material effects or modulation depth and frequency from measured violations of reciprocity. For instance, there have been reports (anecdotal) that the deviations from reciprocity for multiple point-to-point ultrasonic measurements in concrete undergoing fatigue loading correlate with the duration and intensity of loading. Proposed mechanisms include increased nonlinearity due to the growth of microcracks. Unambiguous identification of nonreciprocity requires careful prior calibration of the acoustoelectric equipment, but if implemented properly, nonreciprocal data measurement could become a useful tool for ultrasonic non-destructive evaluation, a multibillion-dollar industry. This is but one practical example of the implications of the value in understanding and designing nonreciprocal acoustic and elastic materials. This Review outlines many strategies, in theory and in practice, for achieving nonreciprocity of acoustic and elastic wave motion in materials. However, this is a highly active and rich area of study, and we anticipate substantial advances to be made in the years to come. There is certainly a need for greater one-way transmission efficiency, frequency bandwidth and amplitude independence, and research with that focus will certainly be built upon the scientific principles and advances reviewed here.

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Author contributions

H.N., B.Y., M.R. and R.F. provided initial drafts of portions of the manuscript. All authors subsequently integrated, reviewed and revised the full manuscript. M.R.H. and G.H. coordinated manuscript writing and organization.

Competing interests

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