An elastic metamaterial with simultaneously negative mass density and bulk modulus

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In this letter, an elastic metamaterial which exhibits simultaneously negative effective mass density and bulk modulus is presented with a single unit structure made of solid materials. The double-negative properties are achieved through a chiral microstructure that is capable of producing simultaneous translational and rotational resonances. The negative effective mass density and effective bulk modulus are numerically determined and confirmed by the analysis of wave propagation. The left-handed wave propagation property of this metamaterial is demonstrated by the negative refraction of acoustic waves. © 2011 American Institute of Physics.

In 1967, Veselago theoretically investigated a visionary material with simultaneously negative permittivity and permeability. This material is termed as a left-handed material (LHM). The concept did not become a reality until 2001 when Shelby et al. proposed designs of structured materials with microstructures of metallic wires and split-ring resonators. The LHM leads to many unusual characteristics: negative refraction, reversed Doppler effect, reversed Cerenkov radiation, and superlensing for applications in information and communication technologies.

In the same time, there has been also a great interest to design elastic/acoustic (EA) metamaterials. The EA metamaterial was experimentally demonstrated through the localized resonance structure constructed by coating a heavy sphere with soft silicone rubber which is then encased in epoxy. On the other hand, Fang et al. designed an acoustic metamaterial by using an array of subwavelength Helmholtz resonators. Bipolar and monopolar local resonances were later proved to be the essential wave mechanisms for producing negative effective mass density (NMD) and negative effective bulk modulus (NBM) of the metamaterial, respectively. Various EA metamaterial designs of solid and fluid unit mixture were proposed to obtain simultaneously NMD and NBM. However, the aforementioned designs all require the combined use of solid and fluid media and are difficult to fabricate. Furthermore, the resulting frequency band for double-negative properties is usually narrow. Recently an experiment of ultrasonic focusing was reported by Zhang et al., who construct a fluid LHM using a planar network of subwavelength Helmholtz resonators. However, a practical double-negative EA metamaterial made purely from solid materials is still absent.

In this letter, we propose a double-negative EA metamaterial which can be achieved with solid materials. In the proposed EA metamaterial, the effective NBM is achieved by a chiral microstructure. To illustrate the concept, we consider a one-dimensional (1D) chiral mass-spring unit shown in Fig. 1(a). Four mass-less springs and a rigid disk with rotational inertia \( I \) are pin-connected. The two springs with spring constant \( k_2 \) are tangential to the disk with an angle \( \alpha \). The pin joints A, B, and C are kept in the horizontal axis. The rotation of the disk is induced by equally applied force \( F \). For infinitesimal deformation, the force in the spring \( k_2 \) is given by \( f_2=k_2(R\theta-x\cos\alpha) \). The dynamic motion of the disk of radius \( R \) is then governed by \( 2f_2R=-I\dddot{\theta}/\partial t^2 \). The balance of forces at pin joint A gives \( F=k_1\ddot{x}-f_2\cos\alpha \). By assuming time-harmonic quantities, i.e., \( (F, x, \theta)=(\hat{F}, \hat{x}, \hat{\theta})e^{i\omega t} \), eliminating \( \theta \) and defining the dynamic effective stiffness \( k_{\text{eff}} \) of the system by \( \hat{F}=k_{\text{eff}}(2\hat{x}) \), we have

\[
k_{\text{eff}} = \frac{k_1}{2} + \frac{k_2(\cos\alpha)^2}{2}\left(1 - \frac{\omega_0^2}{\omega_0^2 - \omega^2}\right),
\]

where \( \omega_0=\sqrt{2k_2R^2/I} \) is the rotational resonance frequency. Equation (1) reveals that \( k_{\text{eff}} \) becomes negative in the fre-
frequency range \( \sqrt{k_1/(k_1+k_2 \cos^2 \alpha)} < \omega/\omega_0 < 1 \). Since the translation resonance of the disk can be used to generate the effect of NMD, it is conceivable that the chiral microstructure may produce both NMD and NBM at a common frequency. Based on the discrete model of Fig. 1(a), a practical continuum two-dimensional (2D) metamaterial is proposed and its unit cell (representative volume element) is depicted in Fig. 1(b). The unit cell is based on the 2D analogy of the well-known three-component sonic crystal, i.e., soft-coated heavy cylinder core embedded in a matrix. A number of \( (n_x) \) slots with width \( t_x \) are cut out from the coating material. The slots are equispaced in azimuth and oriented at an angle \( \theta_x \) with respect to the radial direction. The periodic structure is arranged in a triangular lattice with lattice constant \( a \). It is known that the dispersion relations for elastic metamaterials with triangular lattices are isotropic near the \( \Gamma \) point in the wave vector space. The unit structure lacks any planes of mirror symmetry and, thus, it is said to be chiral. It is known that the traditional three-component metamaterial without chirality gives NMD because of the translational resonance of the core. However, the rotational resonance can never be excited because of the mirror symmetry. By introducing chirality, rotational resonance may be coupled to produce the overall dilatation of the unit cell. As a result, NBM could be obtained. With an appropriate design of the unit cell, we anticipate to achieve both types of resonance in the overlapped frequency range and hence the double negativity.

Due to the geometric complexity, the use of analytical methods for solving the dynamic response of the 2D EA metamaterial is not practicable. Instead, a numerical method is suggested to determine the effective properties of the proposed 2D metamaterial. Under the long wavelength assumption, the global stress, strain, resultant force and acceleration of the unit cell can be obtained by averaging local quantities on the external boundary as

\[
\Sigma_{\alpha\beta} = \frac{1}{V} \int_{\partial V} \sigma_{\alpha\gamma} \gamma\beta \rho d\gamma, \quad E_{\alpha\beta} = \frac{1}{2V} \int_{\partial V} (u_{\alpha\gamma} s_{\beta\gamma} + u_{\beta\gamma} s_{\alpha\gamma}),
\]

\[
F_{\alpha} = \frac{1}{V} \int_{\partial V} \sigma_{\alpha\beta} d\rho_{\beta}, \quad \ddot{U}_{\alpha} = \frac{1}{S} \int_{\partial V} \ddot{u}_{\alpha} d\sigma,
\]

respectively, where \( \alpha, \beta, \gamma = 1, 2 \), \( \sigma_{\alpha\beta}, u_{\alpha\gamma} \), and \( \ddot{u}_{\alpha} \) are the local stress, displacement, and acceleration fields, respectively, \( d\sigma = n_a d\sigma \) with \( n_a \) being the boundary unit normal, \( x_a \) is position vector, and \( V \) and \( \partial V \) denote unit cell’s volume and external boundary. Considering the macroscopic isotropy, the effective bulk, shear modulus and mass density for the 2D problem can be defined as

\[
K_{\text{eff}} = \frac{1}{2} \Sigma_{\alpha\beta} F_{\alpha\beta}, \quad \mu_{\text{eff}} = \frac{1}{2} \Sigma_{\alpha\beta} \cdot E_{\alpha\beta}, \quad \rho_{\text{eff}} = F_{\alpha} \ddot{U}_{\alpha},
\]

where \( \Sigma_{\alpha\beta} \) and \( E_{\alpha\beta} \) mean the deviatoric parts of the global stress and strain, respectively. The local displacement field can be assumed as \( \ddot{u}_{\alpha} = \ddot{u}_{\alpha}^0 + \ddot{E}_{\alpha\beta} x_{\beta} \) where the displacement field \( \ddot{u}_{\alpha} \) is compatible with a given macrostrain tensor \( \ddot{E}_{\alpha\beta} \) plus a given rigid translation \( u_{\alpha}^0 \). To obtain the local fields, the time-harmonic displacement is applied on the unit cell’s boundary \( \partial V \) as \( u_{\alpha}(x,t) = \ddot{u}_{\alpha} x_{\beta} e^{i\omega t} \). The problem is solved by using the finite element method (FEM) with frequency swept over the interested range, from which the global stress and force can be determined based on Eq. (2). In

![FIG. 2. (Color online) (a) Effective dynamic density, (b) bulk modulus, and (c) band structure of the elastic metamaterial. The predicted overlapped NMD and NBM frequency range matches the negative pass band in the band diagram which is highlighted by dashed line.](image-url)

the numerical example, epoxy, lead, and low-density polyethylene are chosen to be the materials of matrix, core, and coating, respectively. The core is 5.6 mm in diameter and the coating thickness is 0.7 mm. The triangular lattice constant is \( a=10.75 \) mm. The slot parameters are \( n_x=12, \ t_x=0.4 \) mm, and \( \theta_x=56^\circ \). Figure 2(a) shows normalized effective mass density \( \rho_{\text{eff}}/\rho_0 \) as a function of wave frequency, where \( \rho_0 \) is the density of the epoxy. It is seen that the NMD occurs in the range of 9.51–21.54 kHz. Figure 2(b) shows the normalized bulk modulus \( K_{\text{eff}}/\rho_m \) and shear modulus \( \mu_{\text{eff}}/\mu_m \) as a function of frequency, where \( \mu_m \) is the shear modulus of the epoxy. It is of interest to note that \( K_{\text{eff}} \) becomes negative in the frequency range of 14.08–14.72 kHz while \( \mu_{\text{eff}} \) is always positive. Further, in the range of 14.30–14.72 kHz both effective longitudinal modulus \( E_{\text{eff}}=K_{\text{eff}}+\mu_{\text{eff}} \) and effective mass density \( \rho_{\text{eff}} \) become negative. Thus, it is a pass band in which the left-handed wave propagation behavior is implied. To demonstrate this, the band structure calculation is performed by using the FEM in conjunction with the Bloch’s theorem. The wave dispersion relation along \( \Gamma K \) direction is shown in Fig. 2(c). The lattice array and the Brillouin zone are plotted in the inset of Fig. 2(c). A stop band is observed for both longitudinal and transverse wave branches in the frequency range of 9.44–21.58 kHz which perfectly matches the frequency range of the predicted NMD. In addition, a new pass band with negative slope is found in the frequency range of 14.05–14.73 kHz which lies close to the overlapped region of the NMD and negative longitudinal modulus. A discrepancy in band width between the pass band and the double-negative properties prediction may be attributed to the following reasons: (1) the applied boundary condition does not take into account of phase change across the unit cell and is accurate only in the long wave limit, thus the negative band and the double-negative properties matches better at upper bound and (2) for the chiral metamaterial, when the rotational resonance of the core occurs, the global stress would become asymmetric and the classical theory of elasticity cannot fully characterize this material. The higher-order continuum theories may be more suitable for this application.
shown in dashed lines, and the boundary tractions are also depicted by arrows. In Fig. 3(a) (0 kHz), quasistatic K_{eff} is obtained because the frequency is far away from the core rotational resonance frequency. As shown Fig. 3(b), when the frequency (14.5 kHz) approaches the rotational resonance frequency from below, the core rotates in-phase with the overall deformation so that a very large clockwise rotation is generated in conjunction with an expanded unit cell. Such a rotation causes a compression state in the matrix and external boundary resulting in the NBM. In Fig. 3(c), when the frequency (15.2 kHz) approaches the rotational resonance frequency from above, the anticlockwise core rotation enhances the tension state in the matrix and, consequently, a positive peak of dynamic effective modulus occurs.

The most prominent phenomenon related to double-negative metamaterials is the negative refraction. In order to test this, we perform a large scale full wave simulation consists of detailed microstructure using commercial finite element code COMSOL. We take a 30.0° wedged sample which consists of 546 unit cells with a triangular array immersed in water (1000 kg/m³ in density, 1500 m/s in sound speed). A Gaussian acoustic pressure beam is launched in the fluid from the bottom side of the wedge. The acoustic pressure (hydrostatic stress) fields in the system are plotted in Fig. 4(a) at frequency 14.53 kHz, where the effective bulk, shear modulus and mass density of the metamaterial are calculated to be −1.58 GPa, 0.42 GPa, and −1481 kg/m³, respectively. It can be clearly seen that the energy flux of the refraction wave outside of the sample travels on the negative refraction phenomenon can be observed because the frequency is far away from the core rotational resonance frequency. As shown Fig. 3(b), when the frequency (14.5 kHz) approaches the rotational resonance frequency from below, the core rotates in-phase with the overall deformation so that a very large clockwise rotation is generated in conjunction with an expanded unit cell. Such a rotation causes a compression state in the matrix and external boundary resulting in the NBM. In Fig. 3(c), when the frequency (15.2 kHz) approaches the rotational resonance frequency from above, the anticlockwise core rotation enhances the tension state in the matrix and, consequently, a positive peak of dynamic effective modulus occurs.

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